

Uncertain Price Competition in a Duopoly with Heterogeneous Availability

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Abstract

We study the price competition in a duopoly with an arbitrary number of buyers. Each seller can offer multiple units of a commodity depending on the availability of the commodity which is random and may be different for different sellers. Sellers seek to select a price that will be attractive to the buyers and also fetch adequate profits. The selection will in general depend on the number of units available with the seller and also that of its competitor - the seller may only know the statistics of the latter. The setting captures a microgrid network or a secondary spectrum access network in which excess power units and unused spectrum bands constitute the respective commodities of sale. We analyze this price competition as a game, and identify a set of necessary and sufficient properties for the Nash Equilibrium (NE). The properties reveal that sellers randomize their price using probability distributions whose support sets are mutually disjoint and in decreasing order of the number of availability. We prove the existence and uniqueness of a symmetric NE in a symmetric market, and explicitly compute the price distribution in the symmetric NE.

I. INTRODUCTION

In this paper, we study pricing in a duopoly market in presence of uncertainty in competition.

Motivation

We start by describing two example scenarios where uncertainty in competition naturally emerges: micro grid networks and secondary spectrum access networks.

The first example scenario pertains to pricing in micro grids [1]. Distributed generation of power at small on-site stations is a promising alternative to the traditional generation at large stations. Decreasing the loss of transmission by reducing the distance to consumption units, utilizing renewable energy sources, decreasing the risk of blackout, and increasing security are some of the advantages of distributed power generating scheme [2]. A micro grid network is a network of distributed power generating systems connected to local subscribers, and also to the central macro power grid. Each unit equipped with a

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Part of this work was presented in Allerton'12 [22].

distributed power generating system can sell its excess power to local subscribers as well as the macro grid. The amount of power generated by a power generating system is not apriori known and is different for different sellers. Thus, the sellers need to select prices for the excess power they offer for sale, without knowing the number of power units available for sale with their competitors (uncertainty in competition).

Pricing in secondary spectrum access networks [3] is another application of our model. Recent developments in wireless devices have resulted in a significant growth in demand for the radio spectrum. This leads to spectrum congestion. On the other hand, the available radio spectrum is greatly under-utilized [4]. Spectrum congestion and under-utilization have directed researchers to adopt new techniques in order to use the available spectrum more efficiently and to decrease congestion. Secondary spectrum access is an example of these techniques. In these networks, there are two types of users: (i) Primary/licensed users (sellers in our context), who lease a number of frequency bands (channels) directly from the regulator, and (ii) Secondary/unlicensed users (buyers in our context), who lease frequency bands from primary users for a certain amount of time in exchange for money or other types of credit. A secondary user can lease a channel only if it is not in use by the primary user who owns it. The usage of subscribers of primary users is random and different for different primaries. Therefore the primaries need to select prices for the frequency bands they offer for sale, without knowing the number of frequency bands available for sale with their competitors (uncertainty in competition).

The Research Challenges and Goals

We consider a market with two sellers, where each seller offers multiple commodities for sale. We investigate the price selection strategy for sellers in presence of uncertainty in competition using Game Theory [5]. Customers shop around for the lowest available prices. Therefore sellers seek to set prices that will ensure that their commodities are sold and also fetch adequate profit. In our model, a seller is not aware of the number of units available to her competitor before quoting her price. Thus, the competition that each seller faces is uncertain, and different sellers have different number of goods available (heterogeneous availability). Each seller selects the price per unit depending on the number of units she has available for sale, the statistics of the availability process for her competitor, and the demand. In general, each seller chooses her price randomly using different probability distributions for different availability levels. Thus, the strategy of each player is a vector of probability distributions. For instance, if a seller can potentially offer up to three units of commodity, her vector of strategies would be $(\Phi_1(\cdot), \Phi_2(\cdot), \Phi_3(\cdot))$, where $\Phi_i(\cdot)$ is the price selection probability distribution when the seller offers i units.

Due to uncertainty in competition, quoting a high price by a seller enhances the risk of not being able to sell the commodity offered by that seller. On the other hand, although selecting a low price increases

the chance of winning the competition, it also decreases the profit earned by the seller. Therefore, pricing in presence of uncertainty in competition is a risk-reward tradeoff. It is not apriori clear that how offering multiple number of units affects the price selection by sellers. For instance, a seller with a large number of available units may be motivated to quote a low price, since in the event of winning the competition, a small amount of profit per unit would result in a large total profit. On the other hand, a seller maybe enticed to select a high price when the availability is high to significantly increase her overall profit, even at the risk of not being able to sell the available units. We focus on investigating the impact of heterogeneous availability and uncertain competition on the aforementioned risk-reward tradeoff.

Contributions

We start by positioning our work in the context of the existing literature (Section I). We next model the price selection problem as a one-shot non-cooperative game (Section II). The sellers are allowed to have different probability distributions for different availability levels (asymmetric market). Note that since the utility of sellers is not a continuous function of their strategy, classical theorems for existence and uniqueness of NE cannot be used. In Section III, we identify key properties that every NE pricing strategy should satisfy when demand is greater than the maximum possible availability level. The properties reveal that the sellers randomize their price using probability distributions whose support sets are mutually disjoint and in decreasing order of the number of availability. In the context of the aforementioned risk-reward tradeoff, sellers opt for low-risk pricing when they have high availability. In Section IV, we prove that any strategy profile that satisfies the properties listed in Section III constitutes an NE regardless of the relation between the demand and the number of available units. This sufficiency result naturally leads to an algorithm (Appendix C-A) for computing the strategies that satisfy the properties in Section III (If such a strategy exists, it is an NE).

In Section V, we consider a symmetric market and prove that these properties are also necessary conditions for a NE regardless of the relation between the demand and the number of available units. We prove that the symmetric NE exists uniquely, and obtain an algorithm for explicitly computing it. Note that the uniqueness is specific to the symmetric market- our analysis in Appendices C-B and C-C reveals that an asymmetric market allows for multiple Nash equilibria. In Section V-C, the asymptotic behavior of the symmetric NE (when $m \rightarrow \infty$) is investigated through numerical simulations. Results are generalized to the case of random demand in Section VI.

Related Literature

Price competition among different entities has been extensively studied in [6]–[13]. In economics literature as also in the context of specific applications, uncertainty in competition has been investigated

when the availability level is either zero or one [14]–[18]. The strategy profile of each seller consists of only one probability distribution since sellers need to select a price only when they have one unit available for sale. We, however, characterize the Nash equilibrium pricing strategies when sellers have arbitrary and potentially different number of available units for sale (not merely zero or one). In this case, different price selection strategies may be required for different number of available units. Thus, the pricing strategy profile of each seller is a collection of probability distributions, one for each availability value. Therefore both results and proofs are substantially different from previous works.

Another genre of work allows sellers to control the amount of commodities they would generate for sale [19]–[21]. The price that sellers select is derived by a central entity after solving an optimization problem, and is equal for different sellers. We instead consider scenarios where sellers do not control the amount of commodities they produce. In addition, there is no central entity to collect bids in our work: each seller quotes a price depending on the number of available units and her belief about other sellers. Thus, our formulations and results are different from those in this genre.

II. MARKET MODEL AND PROBLEM FORMULATION

A. Market Model

First, we define some preliminary notations. Then sellers' decision and information is described.

1) *Preliminary notations:* We consider a market with two sellers in which each seller owns multiple number of the same commodity and quotes a price per unit. The total demand of the market is d units. For simplicity, the demand is assumed to be deterministic. The generalization to random d is straightforward, and is presented in Section VI.

Buyers prefer the seller who quotes a lower price per unit, and they are equally likely to buy a unit from sellers who select equal prices. Thus, if sellers have a, b units to sell respectively and quote prices of x, y per unit, where $x < y$, then they respectively sell $\min\{a, d\}, \min\{b, (d - a)^+\}$ units, where z^+ denotes $\max\{z, 0\}$. The cost of each transaction is c . Therefore, a seller earns a profit of $i(x - c)$ when she sells i units with price x per unit. Because of regulatory restrictions or because of valuations that buyers associate with purchase of each unit, the price selected by each seller should be bounded by some constant $v > c$, i.e. $x \leq v$. The availability of each seller is random:

Terminology 1. We denote m_k as the maximum possible number of available units of seller k . Let $q_{kj} \in [0, 1]$ be the probability that seller k has $j \in \{0, \dots, m_k\}$ units available, and $\vec{q}_k = (q_{k1}, \dots, q_{km_k})$.

We assume that sellers have zero unit available for sale with positive probability, i.e., $q_{k0} > 0$ for $k \in \{1, 2\}$, and the competition is uncertain, i.e., $q_{ki} < 1$ for $i \in \{0, 1, \dots, m_k\}$ for at least one seller k .

Note that if competition is deterministic for both sellers, then the problem is trivial.

Terminology 2. For each seller k , let \bar{k} denote the other seller, i.e., if $k = 1$ (respectively, $k = 2$), then $\bar{k} = 2$ (respectively $\bar{k} = 1$).

2) *Sellers' decisions and information:* Sellers select their price based on the number of units they offer in the market. Before choosing her price, a seller does not know the number of units of the commodity that her competitor has available for sale and the price per unit her competitor selects. She is however aware of the demand and the distributions for the above quantities. A seller may select her price randomly.

Terminology 3. Let $\Phi_{kj}(\cdot)$ be the probability distribution that the seller $k \in \{1, 2\}$ uses for selecting price per unit when she offers j units. Let \tilde{p}_{kj} and \tilde{v}_{kj} be the infimum and the supremum of the support set¹ of $\Phi_{kj}(\cdot)$. The strategy profile for seller k is $\Theta_k(\cdot) = (\Phi_{k1}(\cdot), \dots, \Phi_{km_k}(\cdot))$.

B. Problem Formulation

Clearly, the number of units a seller sells and her profit are random.

Terminology 4. Let $u_k(\Theta_k(\cdot), \Theta_{\bar{k}}(\cdot))$ denotes the expected profit of seller k when she adopts strategy profile $\Theta_k(\cdot)$ and her competitor adopts $\Theta_{\bar{k}}(\cdot)$.

Definition 1. A Nash equilibrium (NE) is a strategy profile such that no seller can improve her expected profit by unilaterally deviating from her strategy. Therefore, $(\Theta_1^*(\cdot), \Theta_2^*(\cdot))$ is a NE if for each seller k :

$$u_k(\Theta_k^*(\cdot), \Theta_{\bar{k}}^*(\cdot)) \geq u_k(\tilde{\Theta}_k(\cdot), \Theta_{\bar{k}}^*(\cdot)), \quad \forall \quad \tilde{\Theta}_k(\cdot).$$

Terminology 5. With slight abuse of notation, we denote $u_{kl}(x)$ as the expected profit that seller k earns, and $B_{kl}(x)$ as the expected number of units that seller k sells, when she offers l units for sale with price x per unit, respectively (the dependence on the competitor's strategy is implicit in this simplified notation).

$$\text{Clearly, } u_{kl}(x) = B_{kl}(x)(x - c). \quad (1)$$

Terminology 6. Let $e_k = (d - m_{\bar{k}})^+$.

Note that for all $x \leq v$,

$$B_{kl}(x) = l \quad l = 1, \dots, e_k \quad (2)$$

¹The support set of a probability distribution is the smallest closed set such that its complement has probability zero under the distribution function. In other words, if there is another set such that its complement has probability zero, it should be a super set of the support set.

as k will sell all she offers in this case given that the total offering is less than the demand. We would later obtain the expression for $B_{kl}(x)$ under the NE strategy profiles when $l > e_k$.

Definition 2. A price x is said to be the best-response price for seller k when she offers j units if $u_{kj}(x) \geq u_{kj}(a)$ for all $a \in [0, v]$.

Note that a NE-strategy profile selects with positive probability only amongst the best-response prices. Thus, all the elements of support sets are best responses except potentially those on the boundaries (elements of boundaries may not be best responses) if there is a discontinuity in the utility at those points.

We seek to determine the Nash equilibrium strategy profile of sellers. If $m_1 + m_2 \leq d$, since there is no competition between sellers, both sellers offer their units with the monopoly price, v at the NE. We therefore assume that $m_1 + m_2 > d$.

We conclude this section introducing a terminology that is used throughout in the proofs:

Terminology 7. Let $B_{k,l,j}(x) = B_{k,l}(x) - \frac{l}{j}B_{k,j}(x)$ and $A_{k,l,j}(x) = (x - c)B_{k,l,j}(x) = u_{kl}(x) - \frac{l}{j}u_{kj}(x)$.

III. PROPERTIES OF A NE WHEN $d > \max\{m_1, m_2\}$

We investigate the necessary conditions for a strategy to be a NE when $d > \max\{m_1, m_2\}$ (Theorem 1). The focus of this section is on the case that $d > \max\{m_1, m_2\}$, and will explicitly point out whenever we use that assumption.

Theorem 1. A NE must satisfy the following properties when $d > \max\{m_1, m_2\}$,

- 1) For each k , there exists a threshold l_k such that seller k offers price v with probability one if she has $i \in \{1, \dots, l_k\}$ units. This threshold is such that:
 - a) $l_k \in \{e_k, \dots, m_k - 1\}$
 - b) $l_1 + l_2 = d - 1$ or $l_1 + l_2 = d$
- 2) When seller k has $l_k + 1$ units, she uses distribution $\Phi_{k,l_k+1}(\cdot)$
 - a) whose support set is $[\tilde{p}_{k,l_k+1}, v]$,
 - b) which is continuous throughout except possibly at v , and
 - c) has a jump at v for at most one value of $k \in \{1, 2\}$, and size of such a jump is less than 1
- 3) When the availability level is $i \in \{l_k + 2, \dots, m_k\}$, seller k uses distribution $\Phi_{ki}(\cdot)$
 - a) whose support set is $[\tilde{p}_{k,i}, \tilde{p}_{k,i-1}]$,
 - b) which is continuous throughout
 - c) $\tilde{p}_{1,m_k} = \tilde{p}_{2,m_k}$

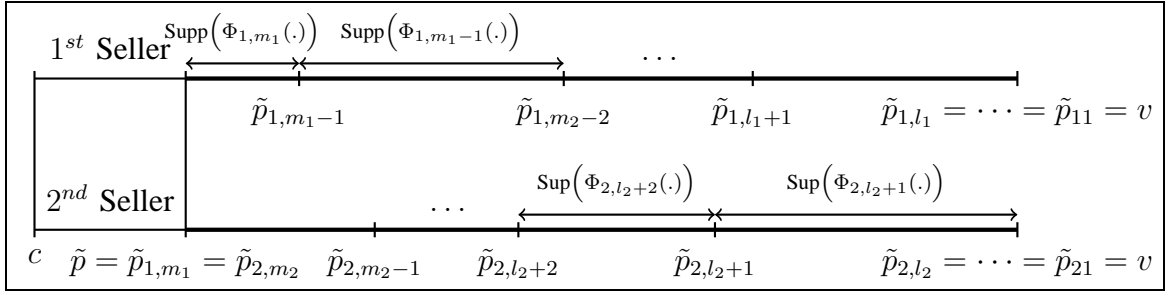


Fig. 1: A schematic example of properties 2 and 3

- 4) The utility of seller k when she offers i units is equal for all prices in the support set of $\Phi_{ki}(\cdot)$, except possibly at price v (if v belongs to her support set).

We prove the above in Section III-G using the following results which we first state and prove later.

- 1) The probability distribution of price, $\Phi_{ki}(x)$ for $i \in \{1, \dots, m_k\}$, is continuous for $x < v$ (Section III-B, Theorem 4).
- 2) The lower bound of prices are equal for the two seller (Section III-C, Theorem 5).
- 3) There is no gap between support sets (Section III-D, Theorem 6).
- 4) Support sets are disjoint barring common boundary points and are in decreasing order of the number of available units for sale (Section III-D, Theorems 7).
- 5) The structure of NE at price v : A seller offers her units with price v , if and only if the number of available units with her is less than or equal to a threshold $l_k \in \{0, 1, \dots, m_k - 1\}$, where $l_1 + l_2 = d$ or $l_1 + l_2 = d - 1$ (Section III-F, Theorem 8).

Figure 1 represents the schematic view of properties 2 and 3 in Theorem 1.

A. Results that we use throughout

Theorem 2. For each i and k , $\Phi_{ki}(c) = 0$.

This result follows directly since prices less than cost c are not chosen by sellers. Theorem 2 therefore rules out jumps at prices $x \leq c$.

Proof: Note that for each i , $u_{ki}(x) \leq 0$ for $x \leq c$. But, since $B_{ki}(x) \geq iq_{k0} > 0$ for all $x \in [0, v]$, $u_{ki}(x) > 0$ for all $x \in (c, v]$. Thus, no price in $[0, c]$ is a best response for a seller. ■

Lemma 1, which we use throughout the paper, rules out jumps at prices higher than c .

Lemma 1. Let the strategy profile of player k be $\Theta_k(\cdot) = (\Phi_{k1}(\cdot), \dots, \Phi_{km_k}(\cdot))$, and $\Phi_{ki}(\cdot)$ have a jump at $x > c$. Then for l such that $l + i > d$, $u_{kl}(x - \epsilon') > u_{kl}(a)$, $\forall a \in [x, \min\{x + \epsilon, v\}]$, and for all sufficiently small but positive ϵ and ϵ' .

We provide the intuition behind the result and defer the proof to the Appendix A. Note that offering

a lower price increases the expected number of units sold by a seller, but decreases the revenue per unit sold. Suppose that a seller k offers i units with price x with a positive probability. Let her competitor \bar{k} have l units available where $l + i > d$; \bar{k} can sell a strictly larger number of units in an expected sense by choosing a price in the left neighborhood of x (eg, $x - \epsilon$) rather than x or in its right neighborhood. In addition the difference is bounded away from zero even as the size of the left neighborhood approaches zero. On the other hand, the difference in the revenue per unit approaches zero as the size of the left neighborhood approaches zero. Therefore, prices in the left neighborhood of x constitute better responses for the seller than x or those in its right neighborhood.

The following theorem fully characterizes the NE when the seller k offers $i \in \{1, \dots, e_k\}$ units.

Theorem 3. $\Phi_{ki}(x)$ selects v with probability 1 and any other prices with probability 0 when $i = 1, \dots, e_k$ for each k .

The proof relies on the fact that if a seller offers less than or equal to e_k units of commodity, she can sell all units regardless of the price she quotes. Therefore v strictly dominates all other prices.

Proof: This statement holds by vacuity if $\max\{m_1, m_2\} \geq d$. Now consider $d > \max\{m_1, m_2\}$. If the seller k offers $i \leq e_k$ units, the total offerings from both sellers are at most d , since the other seller offers at most $m_{\bar{k}}$ units. Thus, the seller k can sell everything it offers with any price x in interval $[0, v]$. Therefore for all $x \in [0, v)$, $u_{ki}(x) = i(x - c) < i(v - c) = u_{ki}(v)$. Thus, no price in $[0, v)$ is a best response. The result follows. ■

B. Continuity of Price Distribution for Price $x < v$

Utilizing Lemma 1, we can prove that the distribution of price is continuous for prices less than v ,

Theorem 4. $\Phi_{ki}(x)$ is continuous for $x < v$.

Proof: If $i \leq e_k$, the property follows from Theorem 3. Now let $i > e_k$. If $x \leq c$, the property follows from Property 2. Now consider $x \in (c, v)$. We use contradiction argument. Suppose $\Phi_{ki}(\cdot)$ has a jump at price $x < v$. Since $i > e_k$, there exists $l \leq m_{\bar{k}}$ such that $l + i > d$. Using lemma 1, we can say that if $\Phi_{ki}(\cdot)$ has a jump at x , for each l such that $l + i > d$, $u_{\bar{k}l}(x - \epsilon') > u_{\bar{k}l}(a)$, where $a \in [x, \min\{x + \epsilon, v\}]$, and for all sufficiently small but positive ϵ and ϵ' . Therefore no price in this interval is a best response for the seller \bar{k} when she offers l units. Therefore $\Phi_{\bar{k}l}(x + \epsilon) = \Phi_{\bar{k}l}(x)$ for all sufficiently small but positive ϵ and all l such that $l > d - i$, i.e. the other seller does not choose any price in $[x, x + \epsilon)$ whenever she offers l units. Knowing this we can say that $B_{ki}(a) = B_{ki}(x)$ for all $a \in [x, x + \epsilon)$ for some $\epsilon > 0$ such that $x + \epsilon \leq v$. Therefore,

$$u_{ki}(x) = (x - c)B_{ki}(x) < (x + \frac{\epsilon}{2} - c)B_{ki}(x + \frac{\epsilon}{2}) = u_{ki}(x + \frac{\epsilon}{2}) \quad (3)$$

Thus, x is not a best response for a seller who offers i units. Hence x is chosen with probability zero, which rules out a jump at x for $\Phi_{ki}(\cdot)$. The theorem follows. \blacksquare

Based on this theorem, the distribution of price is continuous for $x < v$. We will later show that the price distribution has a jump at v for some availabilities.

Based on the above continuity result, the expression for the expected number of units sold for all $x \in [0, v)$ and $l = e_k + 1, \dots, m_k$ is,

$$B_{kl}(x) = l \sum_{i=0}^{d-l} q_{\bar{k}i} + l \sum_{i=d-l+1}^{m_{\bar{k}}} (1 - \Phi_{\bar{k}i}(x)) q_{\bar{k}i} + \sum_{i=d-l+1}^{m_{\bar{k}}} \Phi_{\bar{k}i}(x) q_{\bar{k}i} (d - i) \quad (4)$$

Note that we assumed $d \geq \max\{m_1, m_2\}$ in (4). The first term in the left hand side corresponds to the situation in which the other seller offers at most $d - l$ units. In this case, seller k will sell all l units she offered in the market. The second and the third terms are corresponding to the situation in which the other seller offers more than $d - l$ units with a price higher than and less than x , respectively. If the other seller offers with price higher than x , seller k is able to sell the entire l units. On the other hand, if \bar{k} offers with a price less than x , k will sell $d - l$ units of commodity.

We can now obtain an expression for $u_{kl}(x)$ for $x < v$ from (1), (2), and (4).

C. Sellers Have Equal Lowerbound of Prices

Next, we will prove that the lower bound of prices are equal for the two sellers.

Theorem 5. *The minimum of lower end points of support sets are equal for both sellers. Mathematically,*

$$\tilde{p}_1 = \tilde{p}_2$$

where, $\tilde{p}_k = \min\{\tilde{p}_{ki} : i = 1, \dots, m_k\}$. Furthermore, $\tilde{p}_1 = \tilde{p}_2 < v$ if $d < m_1 + m_2$.

If the lower bound of prices for seller k , i.e. \tilde{p}_k , is lower than that for the other seller, $\tilde{p}_{\bar{k}}$, then k sells equal number of units in an expected sense by choosing \tilde{p}_k as any other price in $(\tilde{p}_k, \tilde{p}_{\bar{k}})$. Using continuity of distributions for prices less than v , we can say that $\tilde{p}_{\bar{k}}$ is a better response than \tilde{p}_k for k , which is a contradiction. The formal proof follows:

Proof: Suppose not. Without loss of generality suppose $\tilde{p}_1 < \tilde{p}_2 \leq v$. Therefore there exists j such that \tilde{p}_1 belongs to the support set of $\Phi_{1j}(\cdot)$. Since player 2 does not offer with any price in the interval

$[\tilde{p}_1, \tilde{p}_2)$, $B_{1j}(\tilde{p}_1) = B_{1j}(\tilde{p}_2^-)$ ². Thus $u_{1j}(\tilde{p}_1) < u_{1j}(\tilde{p}_2^-)$ which contradicts the assumption that \tilde{p}_1 is a best response for the first player when she offers i units of commodity. Therefore, the first part of the property follows.

Suppose $\tilde{p}_1 = \tilde{p}_2 = v$. Thus, both sellers choose the price v with probability 1 regardless of the number of units they have available. Consider seller k . Let $l = m_{\bar{k}}$. Since $m_1 + m_2 > d$, Lemma 1 implies that $u_{\bar{k}m_{\bar{k}}}(v - \epsilon) > u_{\bar{k}m_{\bar{k}}}(v)$. This contradicts the assumption that v is the best response for seller k . The result follows. ■

Terminology 8. Let \tilde{p} denote the minimum of lower end points of prices in the NE, i.e. $\tilde{p}_1 = \tilde{p}_2 = \tilde{p}$.

D. The union of support sets cover $[\tilde{p}, v]$

We show that there does not exist an interval of prices in $[\tilde{p}, v]$ which is eschewed with probability 1 by all sellers. If such an interval existed, the cumulative distribution functions of all sellers would be flat in it, which we rule out below.

Theorem 6. *There does not exist a, b such that $\tilde{p} \leq a < b \leq v$ and $\Phi_{ki}(b) = \Phi_{ki}(a)$ for all $i \in \{e_k + 1, \dots, m_k\}$ and $k = 1, 2$.*

If such a and b exist for seller k , this means that regardless of the number of available units, k does not select any price in the interval (y, z) where $y \leq a$, $z \geq b$, and y is a best response when k has an availability level l . This implies that for the competitor, \bar{k} , price b yields a strictly higher payoff than y . Thus \bar{k} does not select any price in the interval $[y, b)$. This again implies that for seller k , when she offers l units, price b yields a strictly higher payoff than y , which is in contradiction with b being a best response for k when offering l units. The formal proof is as follows:

Proof: Let there be a, b , and k such that $\tilde{p} \leq a < b \leq v$ and $\Phi_{ki}(b) = \Phi_{ki}(a)$ for all i . Consider y such that,

$$y = \sup\{x | x < a, x \in \text{support set of } \Phi_{kl}(\cdot) \text{ for an } l\}$$

Since support sets are closed, y belongs in the support set of $\Phi_{kl}(\cdot)$ for some l . Thus, y is a best response when the availability of player k is l (note that $y < v$).

Note that $\Phi_{ki}(y) = \Phi_{ki}(b)$ for all i . Since $a < b < v$, from Theorem 4 and equation (4), the expected number of units sold for the second seller remains constant for prices in $[y, b]$, regardless of the number of units she offers, i.e. $B_{\bar{k},\cdot}(y) = B_{\bar{k},\cdot}(b)$. Thus, $u_{\bar{k},\cdot}(b) > u_{\bar{k},\cdot}(y)$, and player \bar{k} does not offer any price in the interval $[y, b)$. Therefore $\Phi_{\bar{k},\cdot}(y) = \Phi_{\bar{k},\cdot}(b)$. Since $a < b < v$, from Theorem 4 and equation (4),

² $f(x^-) = \lim_{y \uparrow x} f(y)$

$B_{kl}(y) = B_{kl}(b)$. Thus, $u_{kl}(b) > u_{kl}(y)$. This is in contradiction with y being a best response when the availability of player k is l . Therefore, there does not exist a, b such that $\tilde{p} \leq a < b \leq v$ and $\Phi_{ki}(b) = \Phi_{ki}(a)$ for all $i \in \{1, \dots, m_k\}$ and $k = 1, 2$. Also, note that for $i \in \{1, \dots, e_k\}$, $\Phi_{ki}(b) = \Phi_{ki}(a)$ for $\tilde{p} \leq a < b \leq v$, since support sets for these distributions only contain v . The result follows. \blacksquare

Remark: In all the previous results, we considered $d \geq \max\{m_1, m_2\}$. In the next section, we need to consider that $d > \max\{m_1, m_2\}$.

E. Support Sets Are Mutually Disjoint and in Decreasing Order of the Number of Availabilities

We start with proving a result, Lemma 2, on $A_{k,l,j}(x)$ (defined in Section II, Terminology 7). Note that we use Lemma 2 in subsequent sections as well. We next prove Theorem 7 using this result, which leads to the main results of this section: Corollaries 1 and 2.

First, using (2) and (4).

$$B_{k,l,j}(x) = - \sum_{i=d-l+1}^{d-j} \Phi_{\bar{k}i}(x) q_{\bar{k}i}(i-d+l) + \sum_{i=d-j+1}^{m_{\bar{k}}} \Phi_{\bar{k}i}(x) q_{\bar{k}i}(d-i) \left(1 - \frac{l}{j}\right) \quad (5)$$

Thus, $B_{k,l,j}(\cdot)$ is non increasing and non positive with respect to the price x when $l > j$. Therefore if $l > j$ then $A_{k,l,j}(x)$ is non increasing and non positive with respect to x . Based on the following lemma, $A_{k,l,j}(x)$ is (strictly) decreasing for $v > x \geq \tilde{p}$ and $l > j$ if $d > \max\{m_1, m_2\}$.

Lemma 2. For each seller $k \in \{1, 2\}$ and every l and j , $j < l \leq m_k$, $A_{k,l,j}(x)$ is (strictly) decreasing for $\tilde{p} \leq x < v$ when $d > \max\{m_1, m_2\}$.

Since $A_{k,l,j}(\cdot) = (x - c)B_{k,l,j}(x)$, knowing that $B_{k,l,j}(x)$ is non-increasing, lemma follows if we prove that $B_{k,l,j}(\cdot)$ is negative. We will prove that $\Phi_{km_k}(x)$, which is included in the summation of $B_{k,l,j}(\cdot)$, is positive for $x > \tilde{p}$ and $k \in \{1, 2\}$. In addition, the coefficient of $\Phi_{km_k}(x)$ is negative since $d > \max\{m_1, m_2\}$. Thus, the result follows.

Proof: It is enough to prove that $B_{k,l,j}(x)$ is non-increasing for $x \geq \tilde{p}$ and negative for $x > \tilde{p}$. This yields that $A_{k,l,j}(x) = (x - c)B_{k,l,j}(x)$ is strictly decreasing with respect to x .

Note that in (5), $\Phi_{\bar{k}j}(\cdot)$'s are non-negative and non-increasing since they are probability distributions. In addition, they have negative weights: $-(i - d - l) \leq -1 < 0$, $1 - \frac{l}{j} < 0$, and since $d > \max\{m_1, m_2\}$, $d - i \geq d - m_{\bar{k}} > 0$. Thus $B_{k,l,j}(x)$ is non increasing and non positive with respect to the price x when $l \geq j$. To prove that $B_{k,l,j}(x)$ is negative for $x > \tilde{p}$, since the distributions in (5) have (strictly) negative weights, it is enough to prove that at least one of the $\Phi_{\bar{k}j}(\cdot)$'s is included in the summation of $B_{k,l,j}(\cdot)$ is positive, i.e. not all of them are zero. We will prove that $\Phi_{km_k}(x) > 0$ for $x > \tilde{p}$ and $k \in \{1, 2\}$.

Suppose not and there exists $x > \tilde{p}$ such that $x \leq \tilde{p}_{km_k}$. By Theorem 6, there exists an $\epsilon > 0$ and an availability level $j \neq \{1, \dots, e_k, m_k\}$ such that $[\tilde{p}_{km_k} - \epsilon, \tilde{p}_{km_k}]$ belongs to the support set of $\Phi_{kj}(\cdot)$

and $\tilde{p}_{kj} < \tilde{p}_{km_k}$. Thus $u_{kj}(\tilde{p}_{km_k}) = u_{kj}(\tilde{p}_{km_k} - \epsilon)$. In addition, $B_{k,m_k,j}(x)$ is the weighted summation of $\Phi_{\bar{k}i}(\cdot)$ for $i \in \{e_{\bar{k}} + 1, \dots, m_{\bar{k}}\}$. Theorem 6 implies that \tilde{p}_{kj} belongs to at least one of the support sets of $\Phi_{\bar{k}i}(\cdot)$ for $i \in \{e_{\bar{k}} + 1, \dots, m_{\bar{k}}\}$. The distribution $\Phi_{\bar{k}i}(\cdot)$ is included in the summation of $B_{k,m_k,j}(x)$, and its coefficient is negative. Thus, $A_{k,m_k,j}(x)$ is strictly decreasing with respect to x for $x > \tilde{p}_{kj}$. Thus $A_{k,m_k,j}(\tilde{p}_{km_k} - \epsilon) > A_{k,m_k,j}(\tilde{p}_{km_k})$. Using $u_{kj}(\tilde{p}_{km_k}) = u_{kj}(\tilde{p}_{km_k} - \epsilon)$, we can conclude that $u_{km_k}(\tilde{p}_{km_k}) = u_{km_k,max} < u_{km_k}(\tilde{p}_{km_k} - \epsilon)$. This contradicts with \tilde{p}_{k,m_k} belonging to the support set of $\Phi_{km_k}(\cdot)$. The result follows. \blacksquare

Note that in previous lemma we used $d > \max\{m_1, m_2\}$ to prove that $A_{k,l,j}(x)$ is decreasing for $\tilde{p} \leq x < v$. The following properties characterizes the NE for price less than v .

Theorem 7. *For $k \in \{1, 2\}$, the support set of $\Phi_{kl}(\cdot)$ is a subset of $[\tilde{p}, \tilde{p}_{kj}] \cup [v]$ for all integers $j \in [1, l]$.*

Proof: First note that for $j \in \{1, \dots, e_k\}$ theorem follows, since $\tilde{p}_{kj} = v$ by Theorem 3. Now consider $j > e_k$. Consider support sets of $\Phi_{kj}(\cdot)$, $\Phi_{kl}(\cdot)$, and $j < l$. We will show that $u_{kl}(a) < u_{kl}(\tilde{p}_{kj})$ for all $a \in (\tilde{p}_{kj}, v)$. Thus, no $a \in (\tilde{p}_{kj}, v)$ is a best response for the seller k with availability of l units. Therefore, the support set of $\Phi_{kl}(\cdot)$ is a subset of $[c, \tilde{p}_{kj}] \cup [v]$.

We now complete the proof, by showing that $u_{kl}(a) < u_{kl}(\tilde{p}_{kj})$ for all $a \in (\tilde{p}_{kj}, v)$:

$$u_{kl}(a) - \frac{l}{j}u_{kj}(a) = A_{k,l,j}(a)$$

Since $l > j$ and $\tilde{p} \leq \tilde{p}_{kj} < a < v$, by Lemma 2, $A_{k,l,j}(a)$ is decreasing function of a for $a \in [\tilde{p}_{kj}, v)$. Thus, $A_{k,j}(a) < A_{k,j}(\tilde{p}_{kj})$ for $a \in (\tilde{p}_{kj}, v)$. On the other hand $u_{kj}(a) \leq u_{kj}(\tilde{p}_{kj})$ for all $a > \tilde{p}_{kj}$, since \tilde{p}_{kj} is a best response of a seller with availability j , therefore $u_{kl}(\tilde{p}_{kj}) > u_{kl}(a)$. \blacksquare

Note that, in this stage, since $\Phi_{kl}(\cdot)$ can have a jump at v , we cannot rule out v as a member of the support set of $\Phi_{kl}(\cdot)$.

Corollary 1. *The support sets of $\Phi_{kl}(\cdot)$ and $\Phi_{kj}(\cdot)$ overlap at most at one point in $[\tilde{p}, v)$.*

Proof: Suppose two points x_1 and x_2 , where $x_1 < x_2 < v$, and both points belong to the intersection of the support sets of $\Phi_{kj}(\cdot)$ and $\Phi_{kl}(\cdot)$. Without loss of generality, consider $j < l$. The price $x_2 > \tilde{p}_j$ belongs to the support set of $\Phi_{kl}(\cdot)$, which is a contradiction with Theorem 7. \blacksquare

Corollary 2. *For prices less than v support sets are contiguous (Theorem 6), disjoint (except possibly at one point) (Corollary 1), and in decreasing order of the number of available units for sale (Theorem 7). Thus, there exists an increasing sequence $a_{km_k}, a_{k,m_k-1}, \dots$ of positive real numbers in $(c, v]$ such that the seller k will randomize her price in the interval $[a_{ki}, a_{k,i+1}]$ and possibly $\{v\}$ when she has i units of*

commodity available for sale.

F. The Structure of Nash Equilibrium at Price v

We will investigate the possibility of having a jump at v . First, we prove Lemma 3 which complements previous results by identifying the nature of overlap between $\Phi_{kj}(\cdot)$ and $\Phi_{\bar{k}l}(\cdot)$ for $j \in \{1, \dots, m_k\}$ and $l \in \{1, \dots, m_{\bar{k}}\}$ for prices less than v . Using this lemma, we prove Theorem 8 which is the main result of this section.

Lemma 3. *For every price $\tilde{p} \leq x < v$, it should belong to the support sets $\Phi_{kl}(\cdot)$ and $\Phi_{\bar{k}j}(\cdot)$ such that $l + j > d$.*

A contradiction argument is used to prove the lemma. Assume that there exist x , l , and j such that x belongs to say $\Phi_{kl}(\cdot)$ and $\Phi_{\bar{k}j}(\cdot)$, and $l + j \leq d$. We show that in this case, the expected number of units sold at x and $x + \epsilon$ are equal for seller k when offering l units, i.e. $B_{kl}(x) = B_{kl}(x + \epsilon)$, and subsequently that $u_{kl}(x + \epsilon) > u_{kl}(x)$. Thus x is not a best response for seller k who offers l units, which is a contradiction.

Proof: Suppose not. There exist x , l , and j such that x belongs to say $\Phi_{kl}(\cdot)$ and $\Phi_{\bar{k}j}(\cdot)$, and $l + j \leq d$. We show that there exist $\tilde{j}, \epsilon > 0$ such that $x + \epsilon$ belongs in the support set of $\Phi_{\bar{k}\tilde{j}}(\cdot)$, and subsequently that $u_{kl}(x + \epsilon) > u_{kl}(x)$. Thus x is not a best response for seller k who offers l units which is a contradiction. Consider two cases:

- $x = \tilde{v}_{\bar{k}j}$. Using Corollary 2, x and $x + \epsilon$ belongs to the support set of $\Phi_{\bar{k},j-1}(\cdot)$ when ϵ is small enough. Take $\tilde{j} = j - 1$.
- $x < \tilde{v}_{\bar{k}j}$. If ϵ is small enough, x and $x + \epsilon$ belongs to the support set of $\Phi_{\bar{k}j}(\cdot)$. Take $\tilde{j} = j$.

This also can be observed from Figure 1.

Note that since $l + j \leq d$, $l + \tilde{j} \leq d$. We are going to argue that the expected number of units sold at x and $x + \epsilon$ are equal for seller k , i.e. $B_{kl}(x) = B_{kl}(x + \epsilon)$. To show this, we condition on the number of available units with the seller \bar{k} . If \bar{k} has more than \tilde{j} number of available units, say f , then she will offer with price less than x with probability one. Thus $\tilde{B}_{kl}(x|f) = \tilde{B}_{kl}(x + \epsilon|f) = d - f$ in which $\tilde{B}(\cdot|\cdot)$ is the conditional expected number of units sold. If \bar{k} offers less than \tilde{j} number of units, she will offer with price higher than $x + \epsilon$ with probability one. Thus $\tilde{B}_{kl}(x|f) = \tilde{B}_{kl}(x + \epsilon|f) = l$. If \bar{k} offers \tilde{j} units, since $l + \tilde{j} \leq d$, $\tilde{B}_{kl}(x|\tilde{j}) = \tilde{B}_{kl}(x + \epsilon|\tilde{j}) = l$. Therefore the expected number of units sold at x and $x + \epsilon$ are equal for seller k , and $u_{kl}(x + \epsilon) > u_{kl}(x)$. The proof is complete. ■

Finally, the following theorem investigate the behavior of NE at v .

Terminology 9. l_k is denoted as the highest value such that the seller k with availability $i \in \{1, \dots, l_k\}$ chooses the price v with probability 1; else l_k is defined as 0.

Thus, l_k plays the role of a threshold for player k .

Theorem 8. l_k is such that:

- $l_k \in \{e_k, \dots, m_k - 1\}$
- $l_1 + l_2 = d - 1$ or $l_1 + l_2 = d$

The price distribution $\Phi_{kj}(\cdot)$ does not have a jump at v if $j > l_k + 1$. At most one of the distributions $\Phi_{1,l_1+1}(\cdot)$ and $\Phi_{2,l_2+1}(\cdot)$ can have a jump at v , and size of such a jump is less than 1.

Proof: Theorem 3 shows that $l_k \geq e_k$. We will prove that the threshold l_k should be less than m_k . Note that if seller k has m_k units of availability and she offers her units with a single price v , then $\tilde{p}_k = v$. By Theorems 5 and Theorem 7, the other seller, \bar{k} , offers her units with a single price v regardless of the number of available units. This is a contradiction. The reason is because of Lemma 1. Since $m_1 + m_2 > d$, if $\Phi_{1,m_1}(\cdot)$ has a jump at v , then $u_{2m_2}(v - \epsilon) > u_{2m_2}(v)$, for all sufficiently small but positive ϵ . Thus v is not a best response for the second player when she offers m_2 units, which is a contradiction. Thus $l_k < m_k$. Therefore $l_k \in \{e_k, \dots, m_k - 1\}$.

First, suppose $l_1 + l_2 \geq d + 1$. By lemma 1, v is not a best response for the player k when she offers l_k units, which is a contradiction. Therefore $l_1 + l_2 \leq d$. Next, we will prove that either $l_1 + l_2 = d - 1$ or $l_1 + l_2 = d$. Note that by the definition of l_k , seller k with availability $l_k + 1$ cannot choose the price v with probability 1. Thus using this fact and Corollary 2, the price $x = v - \epsilon$ for $\epsilon > 0$ small enough is in the support sets of $\Phi_{1,l_1+1}(\cdot)$ and $\Phi_{2,l_2+1}(\cdot)$. Thus, by Lemma 3, $l_1 + l_2 \geq d - 1$. Knowing that $l_1 + l_2 \leq d$, the first part of the theorem follows.

Now we should consider the possibility of having a jump at v for $\Phi_{kj}(\cdot)$ for $j \geq l_k + 1$. We will prove that the price distribution does not have a jump at v when seller k offers more than $l_k + 1$ units. Suppose $\Phi_{kj}(\cdot)$ has a jump for $j > l_k + 1$. Note that $j + l_{\bar{k}} > l_k + l_{\bar{k}} + 1 \geq d$. By Lemma 1, v is not a best response for the seller \bar{k} under availability $l_{\bar{k}}$ which contradicts the definition of $l_{\bar{k}}$.

Now consider $l_k + 1$. By definition of l_k such a jump must have a size less than 1, should it exist. We will prove that at most one of the distributions $\Phi_{1,l_1+1}(\cdot)$ and $\Phi_{k,l_2+1}(\cdot)$ can have a jump at v . Suppose not and both have a jump at v . By Lemma 1, since $(l_1 + 1) + (l_2 + 1) > d$, v is not a best response for the player k when she offers $l_k + 1$ units. This is a contradiction. The result follows. ■

Revisiting Equation (4) implies that utility, $u_{ki}(\cdot)$, is continuous not only in interval $[c, v)$, but also at

price v , if $i \leq d - l_{\bar{k}} - 1$. The reason is that for $i \leq d - l_{\bar{k}} - 1$, equation (4) depends only on $\Phi_{\bar{k},j}(\cdot)$ where $j \geq l_{\bar{k}} + 2$, which is continuous at price v based on Theorem 8. If $\Phi_{\bar{k},l_{\bar{k}}+1}(\cdot)$ is continuous at v then $u_{ki}(\cdot)$ is continuous in $[c, v]$ for $i \leq d - l_{\bar{k}}$.

G. Proof of Theorem 1

Proof: Part 1 of Theorem 1 follows from Theorem 8. We now prove part 2. The support set of $\Phi_{k,l_k+1}(\cdot)$ includes at least one $x < v$ from Theorem 8. Thus, Theorems 7 and 6 imply part 2a of this part. Parts 2b and 2c follow from Theorem 4 and Theorem 8, respectively.

We now prove part 3. We start with 3a. Consider $i > l_{k+1}$. From Theorem 8, $\Phi_{k,i}(\cdot)$ does not have a jump at v . From part 2a and Theorem 7, v is not in the supports set of $\Phi_{k,i}(\cdot)$ and $\tilde{v}_{k,i} \leq \tilde{p}_{k,i-1}$. The result can now be proved by induction starting with $i = l_{k+2}$ using the fact that there is no gap between the support sets (Theorem 6). Since v is not in the support set of $\Phi_{k,i}(\cdot)$, part 3b follows from Theorem 4. Part 3c follows from part 3a and Theorem 5

Property 4 follows from the fact that every price in the support set of a NE, except those on the boundaries, should be a best response for a seller. Thus they yield the same utility value. The result follows for the boundary points of the support sets other than v from Theorem 4. ■

IV. ARBITRARY DEMAND

In this section, first we present the sufficiency theorem for $d \geq \max\{m_1, m_2\}$ (Theorem 9). Theorem 9 establishes that a strategy profile which satisfies the mentioned properties in Theorem 1 constitutes an NE when $d \geq \max\{m_1, m_2\}$. Note that unlike Theorem 1, the sufficiency theorem holds even when $d = \max\{m_1, m_2\}$. Thus, the properties in Theorem 1 are both necessary and sufficient conditions for an NE when $d > \max\{m_1, m_2\}$, and only sufficient conditions when $d = \max\{m_1, m_2\}$. The sufficiency theorem naturally leads to an algorithm for computing NE strategy profiles that satisfy the properties in Theorem 1 (Appendix C-A). Any strategy profile obtained by the algorithm constitutes an NE by Theorem 9. In Section IV-B, we argue that the computation of the NE strategies for $d < \max\{m_1, m_2\}$ can be reduced to $d = \max\{m_1, m_2\}$. This completes the entire framework.

A. The Sufficiency Theorem when $d \geq \max\{m_1, m_2\}$

Theorem 9. *Consider a strategy profile that satisfies the properties enumerated in Theorem 1. This strategy profile is a Nash equilibrium when $d \geq \max\{m_1, m_2\}$.*

The proof is presented in Appendix B. In the proof, we use from the fact that $A_{k,l,j}(\cdot)$ is non increasing and non positive when $d \geq \max\{m_1, m_2\}$.

B. Allowing $d \leq \max\{m_1, m_2\}$

Note that all results before equation (4) also hold when $d \leq \max\{m_1, m_2\}$. Thus (4) can be restated by replacing $e_k = d - m_{\bar{k}}$ with $e_k = (d - m_{\bar{k}})^+$:

$$B_{kj}(x) = j \sum_{i=0}^{(d-j)^+} q_{\bar{k}i} + \min\{j, d\} \sum_{i=(d-j)^++1}^{m_{\bar{k}}} (1 - \Phi_{\bar{k}i}(x)) q_{\bar{k}i} + \sum_{i=(d-j)^++1}^{m_{\bar{k}}} \Phi_{\bar{k}i}(x) q_{\bar{k}i} (d - i)^+ \quad (6)$$

Note that if $m_k > d$, the utilities of all number of availability levels $j \geq d$ for player k are equal:

$$u_{kd} = u_{k,d+1} = \dots = u_{km_k} = d \sum_{i=1}^{m_{\bar{k}}} (1 - \Phi_{\bar{k}i}(x)) q_{\bar{k}i} \quad (7)$$

Let $\tilde{q}_{\bar{k}d} = \sum_{i=d}^{m_{\bar{k}}} q_{\bar{k}i}$ and $\tilde{\Phi}_{\bar{k}d}(x) = \sum_{i=d}^{m_{\bar{k}}} \frac{q_{\bar{k}i}}{\tilde{q}_{\bar{k}d}} \Phi_{\bar{k}i}(x)$. Thus, $\tilde{q}_{\bar{k}d}$ is the probability that the number of available units with seller \bar{k} is greater than or equal to d and $\tilde{\Phi}_{\bar{k}d}(x)$ is the average probability distribution associated with selecting the price if seller \bar{k} availability is d or higher. Now, the term $\sum_{i=d}^{m_{\bar{k}}} (1 - \Phi_{\bar{k}i}(x)) q_{\bar{k}i}$ in the expression for $u_{ki}(\cdot)$ in (6) can be replaced by $\tilde{q}_{\bar{k}d}(1 - \tilde{\Phi}_{\bar{k}d}(x))$. Therefore the problem is reduced to finding the structure when $d = \max\{m_1, m_2\}$. In the previous section, it was proved that a strategy profile that satisfies properties in Theorem 1 is a NE when $d = \max\{m_1, m_2\}$. Therefore, a set of equilibria of the game when $d < \max\{m_1, m_2\}$ can be found by defining $\tilde{\Phi}_{kd}(\cdot)$ and using the properties listed in Theorem 1. The distribution of each individual $\Phi_{kj}(\cdot)$ for $j \geq d$ cannot be determined uniquely and is not of significant interest.

In the next section, we focus on the symmetric NE in a symmetric market.

V. THE SYMMETRIC SETTING

We now consider the symmetric setting in which $\vec{q}_1 = \vec{q}_2 = \vec{q}$ (clearly $m_1 = m_2 = m$). In this case, it is natural to consider a symmetric NE, defined as follows,

Definition 3. A NE $(\Theta_1(\cdot), \Theta_2(\cdot))$ is said to be symmetric if $\Theta_1(\cdot) = \Theta_2(\cdot)$.

Thus, when considering symmetric NE, in terminologies like $\Phi(\cdot), \Theta(\cdot), u(\cdot), \tilde{p}$, we drop the index that represents the seller and only retain the index that represents the number of units available for sale. As a special case of the general setting (Sections III and IV), every symmetric NE should satisfy the properties in Theorem 1 when $d > m$, and every strategy profile that satisfies these properties is a NE when $d \geq m$ (Theorem 9). In Section V-A, we extend Theorem 1 to the case of $d = m$. In Section V-B, we will present an algorithm to find symmetric Nash equilibria of the game when $d \geq m$. Using the results in Section IV-B, the algorithm can be extended to $d < m$. In Section V-C, the asymptotic behavior of the symmetric NE (when $m \rightarrow \infty$) is investigated through numerical simulations.

Note that the algorithm reveals that there is only one symmetric strategy profile that satisfies the properties. It follows from Theorems 1 and 9 that a symmetric NE strategy profile exists uniquely when $d \geq m$. More discussions on uniqueness of NE in symmetric and asymmetric scenarios is presented in Appendix C-C.

A. Properties of a Symmetric Nash Equilibrium

Theorem 10. *Let $d = m$. A symmetric NE in the symmetric setting satisfies the properties listed in Theorem 1.*

The proof is technical and is relegated to the Appendix. It implies that properties in Theorem 1 are necessary and sufficient conditions for a symmetric NE when $d \geq m$.

Since NE is symmetric, $l^* = l_1 = l_2$. Thus, $l^* = \frac{d-1}{2}$ or $l^* = \frac{d}{2}$, whichever that is an integer. Since at most one seller can have a jump at v at $l^* + 1$, in a symmetric NE, none of them do. Thus, the properties in Theorem 1 transform to the following in the symmetric context.

- 1) Sellers offer with price v with probability 1, if they have $i \in \{1, \dots, l^*\}$ available units.
- 2) There exists an increasing sequence $a_m, a_{m-1}, \dots, a_{l^*+1}, a_{l^*}$ of positive real numbers in $(c, v]$ with $a_{l^*} = v$ such that each seller randomizes her price in the interval $[a_i, a_{i-1}]$ when she has i units of commodity available for sale for $i \in \{l^* + 1, \dots, m\}$. Thus,
 - a) Support sets are contiguous.
 - b) Support sets are disjoint (except possibly at one point).
 - c) Support sets are in decreasing order of the number of available units for sale.
- 3) Price distribution is continuous for $i \geq l^*$.
- 4) The utility of a seller when she offers i units is equal for all prices in the support set of $\Phi_i(\cdot)$, except possibly at price v (if it belongs to her support set).

B. Algorithm for computing a symmetric NE for the symmetric setting

We will now identify an algorithm to compute strategies that exhibit the properties in the previous subsection. The algorithm reveals that there is only one symmetric strategy profile that satisfies the same. It follows from Theorem 1 and 9 that a symmetric NE strategy profile exists uniquely when $d \geq m$. Note that the algorithm is developed for $d \geq m$. However, with the method presented in Section IV-B, the algorithm can be used to find the equilibrium for $d \leq m$.

Since $\Phi_j(\cdot)$ is completely characterized for $j < \frac{d+1}{2}$, we should characterize $\Phi_j(\cdot)$ for $j \geq \frac{d+1}{2}$, and outline a framework for computing the same. We proceed in an increasing order of j starting with $j = \lceil \frac{d+1}{2} \rceil$. Then moving to $j = \lceil \frac{d+1}{2} \rceil + 1$, etc.

Now, let $\lceil \frac{d+1}{2} \rceil$. Note that $\tilde{v}_{\lceil \frac{d+1}{2} \rceil} = v$ and $\tilde{p}_k = v$ for $k < \lceil \frac{d+1}{2} \rceil$, and $\tilde{v}_k \leq \tilde{p}_{\lceil \frac{d+1}{2} \rceil}$ for $k > \lceil \frac{d+1}{2} \rceil$ (Properties 1 and 2c). Since support sets are ordered (Property 2c) and disjoint (Property 2b), the expression for $u_{\lceil \frac{d+1}{2} \rceil}(x)$ for $x \in [\tilde{p}_{\lceil \frac{d+1}{2} \rceil}, v)$ only depends on $\Phi_{\lceil \frac{d+1}{2} \rceil}(x)$ (Equation (4)). In particular, $u_{\lceil \frac{d+1}{2} \rceil}(v^-)$ can be obtained using the fact that $\Phi_{\lceil \frac{d+1}{2} \rceil}(v^-) = 1$ which follows from the continuity of $\Phi_{\lceil \frac{d+1}{2} \rceil}(\cdot)$ (Properties 3). Next, $u_{\lceil \frac{d+1}{2} \rceil}(x) = u_{\lceil \frac{d+1}{2} \rceil}(v^-)$ for every $x \in [\tilde{p}_{\lceil \frac{d+1}{2} \rceil}, v)$. Thus having $u_{\lceil \frac{d+1}{2} \rceil}(v^-)$, and using continuity, we can find a unique expression for $\Phi_{\lceil \frac{d+1}{2} \rceil}(x)$. Using the fact that $\Phi_{\lceil \frac{d+1}{2} \rceil}(\tilde{p}_{\lceil \frac{d+1}{2} \rceil}) = 0$, $\tilde{p}_{\lceil \frac{d+1}{2} \rceil}$ can be found uniquely.

We now compute the structure of $\Phi_i(\cdot)$, $\forall i > \lceil \frac{d+1}{2} \rceil$ using $\Phi_{i-1}(\cdot), \Phi_{i-2}(\cdot), \dots, \Phi_{\lceil \frac{d+1}{2} \rceil}(\cdot)$ that are computed before $\Phi_i(\cdot)$. We utilize the facts that,

- 1) $\Phi_j(x) = 1$ for $j > i$, $x \in [\tilde{p}_i, \tilde{v}_i]$
- 2) $\Phi_j(x) = 0$ for $j < i$, $x \in [\tilde{p}_i, \tilde{v}_i]$
- 3) $\tilde{v}_i < v$

Thus, from (4),

$$u_i(\tilde{v}_i) = (\tilde{v}_i - c) \left(i \sum_{g=0}^{i-1} q_g + \sum_i^m q_g(d - g) \right) \quad (8)$$

Since $\tilde{v}_i = \tilde{p}_{i-1}$, and \tilde{p}_{i-1} is computed during the computation of $\Phi_{i-1}(\cdot)$, which precedes that of $\Phi_i(\cdot)$, (8) fully specifies $u_i(\tilde{v}_i)$. Furthermore, for $x \in [\tilde{p}_i, \tilde{v}_i]$ the only unknown variable in the expression of $u_i(x)$ is $\Phi_i(x)$. Since $u_i(x) = u_i(v_i)$ for $x \in [\tilde{p}_i, \tilde{v}_i]$,

$$\Phi_i(x) = \frac{i \sum_{g=0}^{i-1} q_g + i q_i + \sum_{g=i+1}^m q_g(d - g) - \frac{u_i(\tilde{v}_i)}{x - c}}{q_i(2i - d)} \quad (9)$$

From (9), $\Phi_i(\tilde{v}_i) = 1$. Thus, for $x \geq \tilde{v}_i$, $\Phi_i(x) = 1$. Now, \tilde{p}_i can be uniquely identified using the fact that $\Phi_i(\tilde{p}_i) = 0$,

$$\tilde{p}_i = c + \frac{(\tilde{v}_i - c) \left(i \sum_{g=0}^{i-1} q_g + \sum_i^m q_g(d - g) \right)}{i \sum_{g=0}^{i-1} q_g + i q_i + \sum_{g=i+1}^m q_g(d - g)} \quad (10)$$

Therefore $\Phi_i(x) = 0$ for $x \leq \tilde{p}_i$. Clearly, $\Phi_i(\cdot)$ has been characterized uniquely. Note that the denominator of (10) is positive since $d \geq m$ and $q_m < 1$ (uncertainty assumption in Section II). In addition, $\tilde{p}_i > c$. This is because of the fact that the second term of RHS of (10) is positive (as $d \geq m$ and $q_i \geq 0$ for all i).

We now prove that $\Phi_i(\cdot)$ is a valid probability distribution. Clearly, $\Phi_i(\cdot)$ is continuous. Note that in (9) for $x \in [\tilde{p}_i, \tilde{v}_i]$, by increasing x , the term $\frac{u_i(\tilde{v}_i)}{x - c}$ will strictly decrease (since $u_i(\tilde{v}_i) > 0$), and we can say that $\Phi_i(x)$ is strictly increasing. Also, $\Phi_i(\tilde{p}_i) = 0$ and $\Phi_i(\tilde{v}_i) = 1$. Thus, $0 \leq \Phi_i(x) \leq 1$ for $x \in [\tilde{p}_i, \tilde{v}_i]$. Therefore, $\Phi_i(\cdot)$ is non-decreasing and assumes values in $[0, 1]$ for all x . The claim follows. Thus we

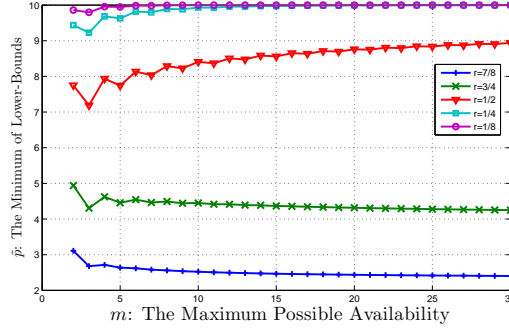


Fig. 2: \tilde{p} versus m for when availability level is binomial with probability p and demand is m

have uniquely identified a symmetric strategy that satisfies the properties required by a Nash equilibrium.

C. The Asymptotic Behavior

The focus of this section is on the asymptotic behavior of the symmetric NE of a symmetric duopoly market when the number of available units with a seller increases to infinity. In asymptotic scenario, many of availability probability distributions that arise naturally concentrate around the mean. Thus, $q_k \rightarrow 0$, when k is far from the mean. First, we show that the length of the support set for availability of k units approaches zero as $q_k \rightarrow 0$: From equation (10),

$$\tilde{p}_i = c + \frac{(\tilde{p}_{i-1} - c)(i \sum_{g=0}^{i-1} q_g + \sum_{g=i}^m q_g(d-g))}{i \sum_{g=0}^i q_g + \sum_{g=i+1}^m q_g(d-g)} = \tilde{p}_{i-1} + (\tilde{p}_{i-1} - c) \frac{q_i(d-2i)}{i \sum_{g=0}^i q_g + \sum_{g=i+1}^m q_g(d-g)}$$

It is immediate that if $q_i \rightarrow 0$, then $\tilde{p}_i \rightarrow \tilde{p}_{i-1}$ ³. This implies that the length of the support set for the availability level i units approaches zero.

We investigate the asymptotic behavior using numerical simulations when the availability of each seller follows a binomial distribution $(m, r < 1)$. With this distribution, as $m \rightarrow \infty$, the binomial distribution can be approximated by a normal distribution with mean mr and variance $mr(1-r)$. Thus $m \rightarrow \infty$ yields that $\tilde{p}_i \rightarrow \tilde{p}_{i-1}$ when $|i - mr|$ is large enough. In other words, the length of the support set for the availability level i units approaches zero if i is far from the mean. Other parameters are considered to be $v = 10$, $c = 1$, and $d = m$.

In Figure 2, the value of \tilde{p} , i.e. the lowest lower-bound is plotted versus m , i.e. the highest possible level of availability. As you can see, the larger the probability r , the smaller \tilde{p} . Note that when r is large, the seller is more likely to offer with higher levels of availability. Therefore the competition is more intense. In addition, when m is increased, the distribution of the random variable q , i.e. the availability random variable, concentrates around the mean, mr . If $r > \frac{1}{2}$, when a seller offers $k = mr$, knowing

³Note that the denominator is positive since $d \geq m$, and we assume uncertainty in competition, i.e. $q_m < 1$.

that the other seller offers $mr > \frac{m}{2}$ with positive probability, she will offer price less than v (note that $d = m$). Furthermore, the higher m , the more intense the competition, and consequently \tilde{p} is decreasing. On the other hand, when $r \leq \frac{1}{2}$, if a seller offers around mr units, there is no competition between sellers knowing that $2mr \leq d = m$. Furthermore, the availability probability q_k , when k is far from mr , tends to zero when m is large. Thus the associated support sets shrink to zero. This justifies the increasing behavior of \tilde{p} . We notice oscillation in the figure, since m alternates between odd and even.

VI. RANDOM DEMAND

We have so far assumed that the demand d is deterministic. In this section, we will generalize the results to a random demand, D . Let r_d denotes the probability that the demand is d , $B_{kld}(x)$ be the expected number of units that seller k sells if she offers l units for sale and quotes x as the price per unit when the total demand is d , and $u_{kld}(x)$ be the expected utility in this case. Clearly,

$$u_{kl}(x) = \sum_d r_d u_{kld}(x) = \sum_d r_d B_{kld}(x)(x - c)$$

We introduce $\underline{d} = \min\{d : d > 0 \text{ and } r_d > 0\}$. Utilizing similar proofs, we can show that all the previous results about the structure of NE (including necessary and sufficient conditions) are valid for the random demand, once d is replaced with \underline{d} . This is but expected as each seller now chooses her price knowing that she is assured of an overall demand of at least \underline{d} (instead of d in the deterministic demand case). Algorithms similar to those in the deterministic case can be developed for computation of the NE in both symmetric and general cases.

VII. CONCLUSION

We investigated price competition in a duopoly market with uncertain competition when different sellers may have different number of units available for sale. We modelled the interactions among the sellers as a non-cooperative game and listed a set of properties that are sufficient conditions for a strategy profile to be a NE. We proved that these properties are also necessary conditions for a NE in a symmetric market, or for some values of demand values in an asymmetric market. We showed that there exists a unique symmetric NE and presented an algorithm for computing the same. A direction for future work is to generalize the results to an oligopoly market.

APPENDIX A

PROOF OF LEMMA 1

Proof: First consider the tuple $\langle l, y \rangle$ associated with the seller \bar{k} in which the first element is the number of units she offers and the second one is the price she chooses. We introduce $D_{kl}^{(1)}(y, i, x)$

as the expected number of units sold by the seller k who wants to offer l units with price y when her competitor's tuple $\langle g, z \rangle \neq \langle i, x \rangle$, and $D_{kl}^{(2)}(y, i, x)$ as the expected number of units sold by the seller who wants to offer l units with price y when her competitor's tuple $\langle g, z \rangle = \langle i, x \rangle$. The expected number of units sold by a seller can be written as,

$$B_{kl}(y) = D_{kl}^{(1)}(y, i, x)Pr\{\langle g, z \rangle \neq \langle i, x \rangle\} + D_{kl}^{(2)}(y, i, x)Pr\{\langle g, z \rangle = \langle i, x \rangle\}$$

Note that $D_{kl}^{(1)}(a, i, x) \leq D_{kl}^{(1)}(x, i, x)$ and $D_{kl}^{(2)}(a, i, x) \leq D_{kl}^{(2)}(x, i, x)$ for $a \geq x$ because the number of units a seller sells is a non-increasing function of her price for any given amounts offered by both sellers and any given price chosen by the competitor. Thus $B_{kl}(a) \leq B_{kl}(x)$. In addition,

$$\begin{aligned} B_{kl}(x - \epsilon') - B_{kl}(x) &= (D_{kl}^{(1)}(x - \epsilon', i, x) - D_{kl}^{(1)}(x, i, x))Pr\{\langle g, z \rangle \neq \langle i, x \rangle\} \\ &\quad + (D_{kl}^{(2)}(x - \epsilon', i, x) - D_{kl}^{(2)}(x, i, x))Pr\{\langle g, z \rangle = \langle i, x \rangle\} \end{aligned} \quad (11)$$

As we discussed $D_{kl}^{(1)}(x, i, x) \leq D_{kl}^{(1)}(x - \epsilon', i, x)$. For $D_{kl}^{(2)}(x, i, x)$, we should consider ties. Since each buyer is equally likely to buy a unit from both sellers if both select equal prices, we can say that $D_{kl}^{(2)}(x, i, x) = l \frac{d}{i+l} < l$ (since $i + l > d$) and $D_{kl}^{(2)}(x - \epsilon, i, x) = l$. Note that $Pr\{\text{other seller's tuple } \langle g, z \rangle = \langle i, x \rangle\} = q_i \times \text{Jump Size of } \Phi_{ki}(\cdot) \text{ at } x$. Thus, for all positive ϵ' , RHS of (11) is greater than or equal to $\theta(x)$, where $\theta(x)$ is a positive number that does not depend on ϵ . Therefore since $B_{kl}(a) \leq B_{kl}(x)$, $\forall a \geq x$, $B_{kl}(x - \epsilon') \geq B_{kl}(a) + \theta(x)$, for all $a \geq x$. Thus,

$$u_{kl}(x - \epsilon') - u_{kl}(a) \geq (x - \epsilon' - a)B_{kl}(a) + \theta(x)(x - \epsilon' - c)$$

Since $x > c$, for all sufficiently small ϵ' , $x - \epsilon' - c > 0$. In addition, since $a \leq x + \epsilon$ by the statement of the lemma, the lowest value for $x - \epsilon' - a$ is $-\epsilon - \epsilon'$, and $B_{kl}(a) \leq l$. Therefore $(x - \epsilon' - a)B_{kl}(a) + \theta(x)(x - \epsilon' - c) \geq (-\epsilon - \epsilon')l + \theta(x)$. Therefore, for all sufficiently small but positive ϵ and ϵ' ,

$$u_{kl}(x - \epsilon') > u_{kl}(a) \quad a \in [x, \min\{x + \epsilon, v\}]$$

■

APPENDIX B

PROOF OF THEOREM 9

Proof: The goal is to show that for each i and k all $x \in [\tilde{p}_{ki}, \tilde{v}_{ki})$ constitutes a best response for the seller k who offers i units. That is, for each $x \in [\tilde{p}_{ki}, \tilde{v}_{ki})$ and for all y , $u_{ki}(x) \geq u_{ki}(y)$. In addition, if $\Phi_{ki}(\cdot)$ associates positive probability with \tilde{v}_{ki} , then $u_{ki}(\tilde{v}_{ki}) \geq u_{ki}(y)$ for all y , i.e., v_{ki} is a best response when the seller k offers i units.

The distributions, $\Phi_{ki}(\cdot)$'s, should satisfy Theorem 4. Thus, equations (4) and (5) holds for $x < v$, and $A_{k,l,j}(x)$ is non increasing and non positive with respect to x for $l > j > e_{\bar{k}}$. We consider the case $j \leq e_{\bar{k}}$ here. Then $B_{k,j}(x) = j$ and $B_{k,l,j}(x) = B_{k,l}(x) - l$. Note that the expected number of units $B_{k,l}(x)$ sold at price x when l units are offered is a non-increasing function of x and $B_{k,l}(x) \leq l$. Thus, $B_{k,l,j}(x)$ and therefore $A_{k,l,j}(x)$ is non increasing and non positive with respect to x for $l > j$ regardless of how j compares with $e_{\bar{k}}$.

Consider $x < \tilde{p}$. $u_{ki}(x) \leq i(x - c) < i(\tilde{p} - c) = u_{ki}(\tilde{p})$. The last equality follows from (4), since $\Phi_{\bar{k}j}(\tilde{p}) = 0$ for all j . Therefore we consider $x \geq \tilde{p}$ throughout the proof.

Suppose $l_k \in \{0, 1, \dots, m_k - 1\}$ in Theorem 8 is fixed, that is, the seller k offers her units with a single price v when she has $i \leq l_k$ available units, and with price less than or equal to v when $i > l_k$. We first start with $i \geq l_k + 1$. From the assumption in Theorem 9, we know that $u_{ki}(x) = u_{ki}(y)$ for any x, y in the interior of the support set of $\Phi_{ki}(\cdot)$, the support set of $\Phi_{ki}(\cdot)$ is $[\tilde{p}_{ki}, \tilde{v}_{ki}]$, $\Phi_{ki}(\cdot)$ is continuous at all $x < v$, $\tilde{v}_{ki} < v$ for $i > l_k + 1$, and $\tilde{v}_{ki} = v$ for $i = l_k + 1$. Thus, if $i > l_k + 1$ $u_{ki}(x) = u_{ki}(y)$ for all $x, y \in [\tilde{p}_{ki}, \tilde{v}_{ki}]$, and for $i = l_k + 1$, $u_{ki}(x) = u_{ki}(y)$ for all $x, y \in [\tilde{p}_{ki}, \tilde{v}_{ki}]$. We consider the last case in detail. Here, $\tilde{v}_{ki} = v$. If \bar{k} has a jump at v when she offers $l_{\bar{k}} + 1$ units, by Lemma 1, $u_{ki}(v) < u_{ki}(v - \epsilon)$ for arbitrary small but positive ϵ .⁴ If not, using equation (4) and continuity of the price distributions included in that equation, it follows that $u_{ki}(v) = u_{ki}(\tilde{p}_{ki})$. Thus, we only need to prove that for all x , $u_{ki}(\tilde{p}_{ki}) \geq u_{ki}(x)$. We do so by separately considering three cases: 1. $i \geq l_k + 1$ and $x \in [\tilde{p}, \tilde{p}_{ki}]$ 2. $i \geq l_k + 1$ and $x \in (\tilde{v}_{ki}, v]$ 3. $i \leq l_k$.

1) $i \geq l_k + 1$ and $x \in [\tilde{p}, \tilde{p}_{ki}]$: The claim follows by vacuity for $i = m_k$. We therefore consider $i < m_k$. Since $\tilde{v}_{kj} = \tilde{p}_{k,j-1}$ for $j \geq l_k + 1$, any such x is in $[\tilde{p}_{kg}, \tilde{p}_{k,g-1})$ for some $g > i$. We prove this claim by induction on g , starting with the base case of $g = i + 1$. For $x \in [\tilde{p}_{k,i+1}, \tilde{p}_{ki})$,

$$\begin{aligned} u_{k,i+1}(x) - \frac{i+1}{i} u_{ki}(x) &= A_{k,i+1,i}(x) \\ u_{k,i+1}(\tilde{p}_{ki}) - \frac{i+1}{i} u_{ki}(\tilde{p}_{ki}) &= A_{k,i+1,i}(\tilde{p}_{ki}) \\ u_{k,i+1}(x) &= u_{k,i+1}(\tilde{p}_{ki}) \end{aligned}$$

Note that $\tilde{p}_{ki} = \tilde{v}_{k,i+1}$. Subtracting the first and the second equation, we get,

$$\frac{i+1}{i} (u_{ki}(x) - u_{ki}(\tilde{p}_{ki})) = A_{k,i+1,i}(\tilde{p}_{ki}) - A_{k,i+1,i}(x) \leq 0$$

Since $A_{k,l,j}(x)$ is non increasing and non positive with respect to x for $l > j$. Therefore $u_{ki}(x) \leq u_{ki}(\tilde{p}_{ki})$ for $x \in [\tilde{p}_{k,i+1}, \tilde{p}_{ki})$.

⁴Note that Lemma 1 holds for any arbitrary price distributions and not only those that are NE.

We want to prove that $u_{ki}(x) \leq u_{ki}(\tilde{p}_{ki})$ for $x \in [\tilde{p}_{k,g+1}, \tilde{p}_{kg})$, knowing that $u_{ki}(x) \leq u_{ki}(\tilde{p}_{ki})$ for $x \in [\tilde{p}_{kg}, \tilde{p}_{k,g-1})$ and $m_k - 1 \geq g \geq i + 1$ (at the base we had $g = i + 1$).

$$\begin{aligned} u_{k,g+1}(x) - \frac{g+1}{i} u_{ki}(x) &= A_{k,g+1,i}(x) \\ u_{k,g+1}(\tilde{p}_{kg}) - \frac{g+1}{i} u_{ki}(\tilde{p}_{kg}) &= A_{k,g+1,i}(\tilde{p}_{kg}) \\ u_{k,g+1}(x) &= u_{k,g+1}(\tilde{p}_{kg}) \end{aligned}$$

Note that $\tilde{p}_{kg} = \tilde{v}_{k,g+1}$. Subtracting the first and the second equation, we get,

$$\frac{g+1}{i} (u_{ki}(x) - u_{ki}(\tilde{p}_{kg})) = A_{k,g+1,i}(\tilde{p}_{kg}) - A_{k,g+1,i}(x) \leq 0$$

Again we used the fact that $A_{k,l,j}(x)$ is non increasing and non positive with respect to x if $l > j$. Therefore $u_{ki}(x) \leq u_{ki}(\tilde{p}_{kg})$ for $x \in [\tilde{p}_{k,g+1}, \tilde{p}_{kg})$. The induction hypothesis follows: $u_{ki}(x) \leq u_{ki}(\tilde{p}_{ki})$ for $x \in [\tilde{p}_{k,g+1}, \tilde{p}_{kg})$.

2) $i \geq l_k + 1$ and $x \in (\tilde{v}_{ki}, v]$: We have just shown that $u_{ki}(x) \leq u_{ki}(\tilde{p}_{ki})$ for all $x \in [\tilde{p}, \tilde{p}_{ki})$. We now show the same for all $x \in (\tilde{v}_{ki}, v]$. The claim follows by vacuity for $i = l_k + 1$, since $\tilde{v}_{ki} = v$. We therefore consider $i > l_k + 1$. Since $\tilde{v}_{kj} = \tilde{p}_{k,j-1}$ for $l_k + 1 \leq j \leq m_k$, and $\tilde{v}_{k,l_k+1} = v$, any such x is in $(\tilde{p}_{kg}, \tilde{p}_{k,g-1}]$ for some $l_k + 1 < g < i$. We prove this claim by induction on g , starting with the base case of $g = i - 1$. Let $x < v$.

$$\begin{aligned} u_{ki}(x) - \frac{i}{i-1} u_{k,i-1}(x) &= A_{k,i,i-1}(x) \\ u_{ki}(\tilde{p}_{k,i-1}) - \frac{i}{i-1} u_{k,i-1}(\tilde{p}_{k,i-1}) &= A_{k,i,i-1}(\tilde{p}_{k,i-1}) \\ u_{k,i-1}(x) &= u_{k,i-1}(\tilde{p}_{k,i-1}) \end{aligned}$$

Subtracting the first and the second equation, we get,

$$u_{ki}(x) - u_{ki}(\tilde{p}_{k,i-1}) = A_{k,i,i-1}(x) - A_{k,i,i-1}(\tilde{p}_{k,i-1}) \leq 0$$

Note that $A_{k,l,j}(x)$ is non increasing and non positive with respect to x . Therefore $u_{ki}(x) \leq u_{ki}(\tilde{p}_{k,i-1})$ for $x \in (\tilde{p}_{k,i-1}, \tilde{p}_{k,i-2}] \setminus v$. The claim is established in the base case if $\tilde{p}_{k,i-2} < v$. Else, if $\tilde{p}_{k,i-2} = v$, the claim has been shown only for $x \in (\tilde{p}_{k,i-1}, v)$ and we still need to show that $u_{ki}(v) \leq u_{ki}(\tilde{p}_{k,i-1})$, which we proceed to do. Now, let $x = v$. if the seller \bar{k} has a jump when it offers $l_{\bar{k}} + 1$ units, since $i > l_k + 1$, for all sufficiently small but positive ϵ , $u_{ki}(v) < u_{ki}(v - \epsilon)$, and for sufficiently small but positive ϵ , $v - \epsilon \in (\tilde{p}_{k,i-1}, v)$. Since $u_{ki}(v - \epsilon) \leq u_{ki}(\tilde{p}_{k,i-1})$, the base case follows. If not, that is seller \bar{k} does not have a jump when it offers $l_{\bar{k}} + 1$ units, using equation (4) and continuity, we can deduce that $u_{ki}(v) \leq u_{ki}(\tilde{p}_{k,i-1})$. The base case follows.

Now we want to prove that $u_{ki}(x) \leq u_{ki}(\tilde{p}_{k,i-1})$ for $x \in (\tilde{p}_{k,g-1}, \tilde{p}_{k,g-2}]$, knowing that $u_{ki}(x) \leq u_{ki}(\tilde{p}_{k,i-1})$ for $x \in (\tilde{p}_{kg}, \tilde{p}_{k,g-1}]$ and $g \leq i-1$ and $g-1 \geq l_k+1$ (at the base we had $g = i-1$). First, let $x < v$.

$$\begin{aligned} u_{ki}(x) - \frac{i}{g-1}u_{k,g-1}(x) &= A_{k,i,g-1}(x) \\ u_{ki}(\tilde{p}_{k,g-1}) - \frac{i}{g-1}u_{k,g-1}(\tilde{p}_{k,g-1}) &= A_{k,i,g-1}(\tilde{p}_{k,g-1}) \\ u_{k,g-1}(x) &= u_{k,g-1}(\tilde{p}_{k,g-1}) \end{aligned}$$

Subtracting the first and the second equation, we get,

$$u_{ki}(x) - u_{ki}(\tilde{p}_{k,g-1}) = A_{k,i,g-1}(x) - A_{k,i,g-1}(\tilde{p}_{k,g-1}) \leq 0$$

The inequality is because of the fact that $A_{k,l,j}(x)$ is non increasing and non positive with respect to x if $l > j$. Therefore $u_{ki}(x) \leq u_{ki}(\tilde{p}_{k,g-1})$. Furthermore we know from the assumption of induction that $u_{ki}(\tilde{p}_{k,g-1}) \leq u_{ki}(\tilde{p}_{k,i-1})$, thus $u_{ki}(x) \leq u_{ki}(\tilde{p}_{k,i-1})$ for $x \in (\tilde{p}_{k,g-1}, \tilde{p}_{k,g-2}] \setminus v$. We can show that $u_{ki}(v) \leq u_{ki}(\tilde{p}_{k,i-1})$ if $v \in (\tilde{p}_{k,g-1}, \tilde{p}_{k,g-2}]$ exactly as in the base case. The proof that for each $i \geq l_k+1$ each $x \in [\tilde{p}_{ki}, \tilde{v}_{ki})$ is a best response when a seller offers i units is therefore complete.

3) $i \leq l_k$: Now let $i \leq l_k$. Thus, $l_k > 0$. Consider two cases:

- $l_k + l_{\bar{k}} = d-1$. Therefore $i \leq l_k = d - l_{\bar{k}} - 1$. As we previously mentioned, utility $u_{ki}(\cdot)$, is continuous not only in interval $[c, v)$, but also at price v , if $i \leq d - l_{\bar{k}} - 1$. Using (5), and the fact that $A_{k,l,j}(x)$ is non increasing and non positive with respect to x , for $l > j$ and a similar argument to case 1, we can get $u_{ki}(x) \leq u_{ki}(v)$ for all $x \in [\tilde{p}, v)$. The result follows.
- $l_k + l_{\bar{k}} = d$. Therefore $i \leq l_k = d - l_{\bar{k}}$. Note that since $l_k + l_{\bar{k}} + 1 > d$, neither $\Phi_{kl_k+1}(\cdot)$ nor $\Phi_{\bar{k}l_{\bar{k}}+1}(\cdot)$ can have a jump at price v . Therefore $u_{ki}(\cdot)$ is continuous in $[c, v]$. The result follows by a similar argument to that of in the previous case.

■

APPENDIX C

COMPUTATION OF NE STRATEGIES IN AN ASYMMETRIC SETTING

In this section, first we develop a framework to obtain the strategy profiles that satisfy the properties listed in Theorem 1 (Section C-A). Then, we compute these strategies for a simple case of an asymmetric market in which $m_1 = m_2 = d = 3$ (Section C-B). In Section C-C, we show that the system may have multiple Nash equilibria.

A. Framework for computation

In this section, we consider the general case in which the setting may not be symmetric. Therefore the distributions may not be identical and the index representing the seller should be used. In Theorem 9,

it has been proved that the properties listed in Theorem 1 are sufficient properties for a NE whether $d > \{m_1, m_1\}$ or $d = \max\{m_1, m_2\}$. In this section, we use Theorem 1 to obtain a framework to identify a set of Nash equilibria for the game.

First, fix l_1 and l_2 (refer to Property 8). In addition, note that Theorem 1 specifies the ordering of support sets for a seller and not the relative ordering of support sets of the two sellers. Thus, first we will fix an ordering of \tilde{p}_{ki} 's and $\tilde{p}_{\bar{k}j}$'s for $i \in \{l_k + 1, \dots, m_k\}$ and $j \in \{l_{\bar{k}} + 1, \dots, m_{\bar{k}}\}$ such that for seller k and \bar{k} the lower bounds are ordered with a decreasing relation with i and j respectively, and $\tilde{p}_{km_k} = \tilde{p}_{\bar{k}m_{\bar{k}}} = \tilde{p}$. The unknowns that we should determine for a NE are \tilde{p} , $m_k - l_k - 1$ and $m_{\bar{k}} - l_{\bar{k}} - 1$ number of lower bounds other than \tilde{p} for seller k and \bar{k} respectively, and the distribution of price over each support set.

For these particular l_1 , l_2 , and relative ordering of support sets the NE is the solution to the following system of equations,

$$\begin{aligned}
u_{ki}(\tilde{p}_{ki}) &= u_{ki}(\tilde{p}_{k,i-1}^-) & i \in A \\
u_{\bar{k}j}(\tilde{p}_{\bar{k}j}) &= u_{\bar{k}j}(\tilde{p}_{\bar{k},j-1}^-) & j \in A \\
u_{ki}(\tilde{p}_{ki}) &= u_{ki}(\tilde{p}_{\bar{k}j}^-) & i \in A, j : \tilde{p}_{\bar{k}j} \in (\tilde{p}_{ki}, \tilde{p}_{k,i-1}) \\
u_{\bar{k}j}(\tilde{p}_{\bar{k}j}) &= u_{\bar{k}j}(\tilde{p}_{ki}^-) & j \in A, i : \tilde{p}_{ki} \in (\tilde{p}_{\bar{k}j}, \tilde{p}_{\bar{k},j-1}) \\
f_1 f_2 &= 0
\end{aligned} \tag{12}$$

where $A = \{l_k + 1, \dots, m_k\}$. In addition, f_1 and f_2 are the magnitude of jump at v for the first and second seller when they offer $l_k + 1$ and $l_{\bar{k}} + 1$ units, respectively. Note that the first four sets of equations are derived using the fact that the utility of a seller should be equal over the entire support set. The fifth equation ensures that only one seller can have a positive jump at v .

In equation (12), the unknowns are \tilde{p} , $m_1 + m_2 - l_1 - l_2 - 2$ number of lower-bounds other than \tilde{p} , p_1 , p_2 , and $m_1 + m_2 - l_1 - l_2 - 2$ number of probability distributions at some specific points. That is $\Phi_{ki}(\tilde{p}_{\bar{k}j})$ for $i \in \{l_k + 1, \dots, m_k\}$ and j such that $\tilde{p}_{\bar{k}j} \in (\tilde{p}_{ki}, \tilde{p}_{k,i-1})$. By solving the system of equations (12), we can get a candidate NE.

Using the solution, $\Phi_{ki}(\cdot)$ for $k \in \{1, 2\}$ and $i \in \{1, \dots, m_k\}$ can be found. To find the distributions of price for prices less than v , first note that each price $x \in [\tilde{p}, v)$ which is not a lower bound for the support set belongs to exactly one of the support sets of each seller. Therefore, by (4), the expression of utility of player k when it offers i units depends only on x and $\Phi_{\bar{k}j}(x)$, i.e. $u_{ki}(x) = (x - c)G(\Phi_{\bar{k}j}(x))$, where $G(\Phi(\cdot))$ is a decreasing function of $\Phi(\cdot)$, and therefore its inverse exists. On the other hand, the utilities at the lower bounds are obtained from (12) for both sellers. Using Property 4, $\Phi_{\bar{k}j}(x) = G^{-1}(\frac{u_{ki}(\tilde{p}_{\bar{k}j})}{x-c})$. If the resulting $\Phi_{\bar{k}j}(\cdot)$ are valid probability distribution functions, using Theorem 9 we can conclude that

they constitute a NE for the given l_1 , l_2 , and the fixed ordering of lower bounds.

We have shown how to obtain a Nash equilibrium given one exists for a particular choice of l_1 , l_2 , and a relative ordering between the support sets of the two sellers. Note that by changing the choices of the above we can possibly obtain multiple Nash equilibria. In the next sections, we present an example in which there exist at least two equilibria. It is not clear that there always exists an NE; our extensive numerical evaluations have not however lead to an instance where there does not exist one.

B. Example illustration of computation of Nash Equilibria

Consider the case in which each seller offers up to three units and the total demand is exactly three units, i.e. $d = 3$. Without loss of generality we assume that $l_1 \geq l_2$; the strategy profiles in the other case $l_1 < l_2$ can be obtained by swapping the indices of the sellers.

1) First we focus on the case in which $l_1 + l_2 = d - 1 = 2$. In this case, $l_1 = l_2 = 1$ or $l_1 = 2, l_2 = 0$.

If $l_1 = l_2 = 1$, then sellers chooses v with probability 1, if she offers 1 unit of commodity. The price v could be a best response for the player who offers 2 units. Therefore, in order to specify the NE, we should find the lower bounds $\tilde{p}_{13} = \tilde{p}_{23} = \tilde{p}$, \tilde{p}_{12} , \tilde{p}_{22} , jumps at price v (f_1 and f_2), and each distribution $\Phi_{kj}(\cdot)$ for all $k = 1, 2$, and $j = 2, 3$.

First consider the ordering of lower bounds in which $\tilde{p}_{22} \geq \tilde{p}_{12}$ (Figure 3). The system of equations is:

$$u_{13}(\tilde{p}) = u_{13}(\tilde{p}_{12}) \Rightarrow 3(\tilde{p} - c) = (3 - 3q_{23}\Phi_{23}(\tilde{p}_{12}))(\tilde{p}_{12} - c) \quad (13)$$

$$u_{23}(\tilde{p}) = u_{23}(\tilde{p}_{12}) \Rightarrow 3(\tilde{p} - c) = (3 - 3q_{13})(\tilde{p}_{12} - c) \quad (14)$$

$$u_{23}(\tilde{p}) = u_{23}(\tilde{p}_{22}) \Rightarrow 3(\tilde{p} - c) = (3 - 3q_{13} - 2q_{12}\Phi_{12}(\tilde{p}_{22}))(\tilde{p}_{22} - c) \quad (15)$$

$$u_{12}(v^-) = u_{12}(\tilde{p}_{22}) \Rightarrow (v - c)(2q_{20} + 2q_{21} + 2q_{22}f_2 + q_{22}(1 - f_2)) = (\tilde{p}_{22} - c)(2 - 2q_{23}) \quad (16)$$

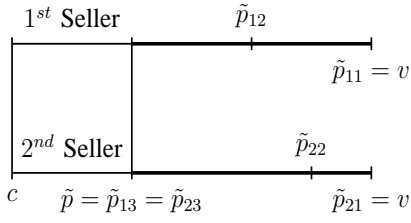
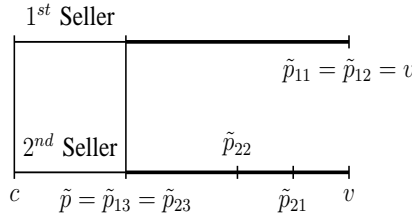
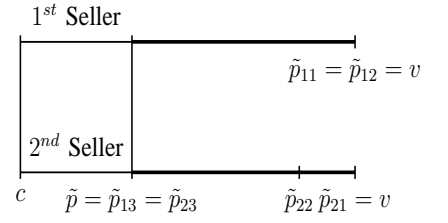
$$u_{12}(v^-) = u_{12}(\tilde{p}_{12}) \Rightarrow (v - c)(2q_{20} + 2q_{21} + 2q_{22}f_2 + q_{22}(1 - f_2)) = (\tilde{p}_{12} - c)(2 - 2q_{23}\Phi_{23}(\tilde{p}_{12})) \quad (17)$$

$$u_{22}(v^-) = u_{22}(\tilde{p}_{22}) \Rightarrow (v - c)(2q_{10} + 2q_{11} + 2q_{12}f_1 + q_{12}(1 - f_1)) = (\tilde{p}_{22} - c)(2 - 2q_{13} - q_{12}\Phi_{12}(\tilde{p}_{22})) \quad (18)$$

$$f_1 f_2 = 0 \quad (\text{At most one seller can have a jump at } v) \quad (19)$$

Using equations (13), (15), (17), and (18), we can find \tilde{p}_{22} as,

$$\begin{aligned} \tilde{p}_{22} &= \frac{(v - c)A}{\frac{1}{2} - \frac{1}{2}q_{13}} + c \\ A &= \left(2q_{10} + 2q_{11} + q_{12}(1 + f_1) - \frac{3}{2}q_{20} - \frac{3}{2}q_{21} - \frac{3}{4}q_{22}(1 + f_2) \right) \end{aligned} \quad (20)$$

Fig. 3: $l_1 = 1$ and $l_2 = 1$ Fig. 4: $l_1 = 2$ and $l_2 = 0$ Fig. 5: $l_1 = 2$ and $l_2 = 1$

On the other hand, from (16),

$$\tilde{p}_{22} = \frac{(v - c)(2q_{20} + 2q_{21} + q_{22}(1 + f_2))}{2 - 2q_{23}} + c \quad (21)$$

The values of \tilde{p}_{22} in (20) and (21) should be equal. Utilizing this and (19),

$$\frac{2f_1q_{12}}{1 - q_{13}} - \frac{1}{2}q_{22}f_2A = (q_{20} + q_{21} + \frac{1}{2}q_{22})A - \frac{4q_{10} + 4q_{11} + 2q_{12}}{1 - q_{13}} = B \quad (22)$$

where $A = \frac{1}{1 - q_{23}} + \frac{3}{1 - q_{13}}$. Therefore,

$$\begin{cases} f_1 = f_2 = 0 & \text{if } B = 0 \\ f_1 > 0 \& f_2 = 0 & \text{if } B > 0 \\ f_2 > 0 \& f_1 = 0 & \text{if } B < 0 \end{cases} \quad (23)$$

Therefore depending on B , f_1 , f_2 , and \tilde{p}_{22} are uniquely determined. Using (18), $\Phi_{12}(\tilde{p}_{22})$ can be derived uniquely,

$$\Phi_{12}(\tilde{p}_{22}) = \frac{1}{q_{12}} \left(2 - 2q_{13} - \frac{v - c}{(\tilde{p}_{22} - c)} (2q_{10} + 2q_{11} + q_{12}(1 + f_1)) \right) \quad (24)$$

By (15), \tilde{p} can be derived uniquely. (14) determines \tilde{p}_{12} uniquely. Equation (13) gives us $\Phi_{23}(\tilde{p}_{12})$ uniquely. However, we should check whether $\Phi_{23}(\tilde{p}_{12})$ and $\Phi_{12}(\tilde{p}_{22})$ are between zero and one or not. If not, then this NE candidate is not valid. The distributions can be found by the process explained previously.

Another possible ordering of lower bounds is when $\tilde{p}_{22} \leq \tilde{p}_{21}$. The system of equations corresponding to this case can be obtained by swapping the index of sellers.

In the case of $l_1 = 2$ and $l_2 = 0$, Figure 4 illustrates a schematic view of the support sets for the unique relative ordering of support sets in this case. Equations can be obtained with a similar approach to the previous case.

2) $l_1 + l_2 = 3 = d$. Note that $l_k = 3$ and $l_{\bar{k}} = 0$ can be ruled out since l_k should be less than $m_k = 3$. Thus, $l_1 = 2$ and $l_2 = 1$ (Figure 5). The approach to find the equilibria is similar to the previous cases.

C. Multiple Nash Equilibria

In Section V, we proved that the symmetric NE exists uniquely. In this section, we show that an asymmetric market allows for multiple Nash equilibria.

Nash equilibria are computed using the above framework with $v = 10$ and $c = 1$ and for different values of \vec{q}_1 and \vec{q}_2 . Some lead to a unique NE and some others to multiple Nash equilibria. For instance, the NE is unique, if

$$\vec{q}_1 = [0.45, 0.1, 0.4, 0.05] \quad \vec{q}_2 = [0.2, 0.2, 0.45, 0.15]$$

In this case, in the only NE strategy profile $l_1 = 1, l_2 = 2, \tilde{p}_{12} = 9.0526, \tilde{p} = 8.65$, and $\Phi_{23}(\tilde{p}_{12}) = 0.3333$, and the second seller has a jump of size 0.625 at price $v = 10$.

But, there are 2 NEs when for instance:

$$\vec{q}_1 = [0.05, 0.1, 0.4, 0.45] \quad \vec{q}_2 = [0.2, 0.2, 0.4, 0.2]$$

In both NE, $l_1 = 2, l_2 = 1$, and $\Phi_{13}(\tilde{p}_{22}) = 0.4444$. In the first NE, $f_2 = 0.06525, f_1 = 0, \tilde{p} = 5.95$, and $\tilde{p}_{22} = 7.1875$. In the second NE, $f_2 = 0, f_1 = 0.7778, \tilde{p} = 5.8$, and $\tilde{p}_{22} = 7$.

APPENDIX D

PROOF OF THEOREM 10

Before going to the proof of Theorem 10, we need to prove some lemmas and theorems. First we prove that $A_{l,j}(x)$ is (strictly) decreasing for $v > x \geq \tilde{p}_{m-1}$ when $d = m$ (Lemma 4). Then, in Lemma 5, we prove that the minimum of the lower end points is the lower end point of $\Phi_m(x)$, i.e., $\tilde{p} = \tilde{p}_m$. Next, using Lemmas 4 and 5, we prove that $\tilde{p}_i \notin [\tilde{p}_m, \tilde{p}_{m-1})$ for $i \in \{1, \dots, m-2\}$. This establishes the ordering for $\Phi_m(\cdot)$ and $\Phi_{m-1}(\cdot)$. After that we proceed to establish the ordering for the remaining support sets $\Phi_j(\cdot)$ for $j \in \{1, \dots, m-2\}$, knowing that for them $\tilde{p}_j \geq \tilde{p}_{m-1}$. A similar result to the Theorem 7 is proved in Theorem 11. Finally, we prove Theorem 10.

Note that a symmetric NE in a symmetric market is considered in this section. Lets define $A_{l,j}(x) = u_l(x) - \frac{l}{j}u_j(x)$. $B_{l,j}(x)$ is defined such that,

$$A_{l,j}(x) = (x - c)B_{l,j}(x)$$

where,

$$B_{l,j}(x) = - \sum_{i=d-l+1}^{d-j} \Phi_i(x)q_i(i-d+l) + \sum_{i=d-j+1}^m \Phi_i(x)q_i(d-i)(1 - \frac{l}{j}) \quad (25)$$

Based on the following lemma, $A_{l,j}(x)$ is (strictly) decreasing for $v > x \geq \tilde{p}_{m-1}$ and $l > j$, when $d = m$.

Lemma 4. For every l and j , $l > j \geq 1$, $A_{l,j}(x)$ is (strictly) decreasing for $v > x \geq \tilde{p}_{m-1}$ when $d = m$.

We argued that $B_{l,j}(\cdot)$ is non increasing and non positive with respect to the price x . To prove that $A_{l,j}(\cdot) = (x - c)B_{l,j}(x)$ is strictly decreasing, it is enough to prove that $B_{l,j}(\cdot)$ is negative. We will prove that $\Phi_{m-1}(x)$ is included in the summation of $B_{l,j}(\cdot)$ and obviously positive for $x > \tilde{p}_{m-1}$. In addition, its coefficient is negative since $d = m > m - 1$. Thus, the result follows.

Proof: It is enough to prove that $B_{l,j}(x)$ is non-increasing for $x \geq \tilde{p}_{m-1}$ and negative for $x > \tilde{p}_{m-1}$ when demand is m . This yields that $A_{l,j}(x) = (x - c)B_{l,j}(x)$ is strictly decreasing with respect to x .

Note that in (25), $\Phi_i(\cdot)$'s are non-negative and non-increasing since they are probability distributions. In addition, they have non-positive weights: $-(i - d - l) \leq -1 < 0$, $1 - \frac{l}{j} < 0$, and $d - i \geq d - m = 0$ (note that $d = m$). Thus $B_{l,j}(x)$ is non increasing and non positive with respect to the price x when $l \geq j$. To prove that $B_{l,j}(x)$ is negative for $x > \tilde{p}_{m-1}$, since $d - (m - 1) = 1 > 0$ and $-(i - d - l) \leq -1 < 0$ (possible coefficients of $\Phi_{m-1}(x)$), it is enough to prove that $\Phi_{m-1}(\cdot)$ is included in the summation of $B_{l,j}(\cdot)$ and it is positive, i.e. $\Phi_{m-1}(x) > 0$ for $x > \tilde{p}_{m-1}$. The later follows from the definition of \tilde{p}_{m-1} .

Now we prove the first statement that is $\Phi_{m-1}(\cdot)$ is included in the summation of $B_{l,j}(\cdot)$. Note that $l > j \geq 1$. Thus $l \geq 2$, and the lowest index of the (25) is $d - l + 1 \leq m - 2 + 1 = m - 1$. The result follows. ■

To prove the ordering and disjoint theorems in the symmetric setting we should alter the proofs. First we will prove that $\tilde{p} = \tilde{p}_m$, i.e. the minimum of lower bounds is the lower bound of $\Phi_m(x)$. Then we will prove that $\tilde{p}_j \notin [\tilde{p}_m, \tilde{p}_{m-1})$ for $j \in \{1, \dots, m - 2\}$. This proves that the next lowest support set is the support set of $\Phi_{m-1}(\cdot)$. After that using Lemma 2 will prove that the support set of $\Phi_l(\cdot)$ for $l < m$ is a subset of $[\tilde{p}_{m-1}, p_j]$ for all integers $j \in [1, l)$. These three all together establishes the ordering.

Lemma 5. $\tilde{p} = \tilde{p}_m$, i.e. the minimum of lower end points is the lower end point of $\Phi_m(x)$.

Proof: Suppose not and there exists $x > \tilde{p}$ such that $x \leq \tilde{p}_m$. By Theorem 6, there exists an $\epsilon > 0$ and an availability level $j \neq m$ such that $[\tilde{p}_m - \epsilon, \tilde{p}_m]$ belongs to the support set of $\Phi_j(\cdot)$ and $\tilde{p}_j < \tilde{p}_m$. Thus $u_j(\tilde{p}_m) = u_j(\tilde{p}_m - \epsilon)$. In addition, $B_{m,j}(x)$ is the weighted summation of $\Phi_i(\cdot)$ for $i \in \{1, \dots, m\}$. Thus, the distribution $\Phi_j(\cdot)$ is included in the summation of $B_{m,j}(x)$, and its coefficient is negative (Note that $d - j > 0$). In addition, $\Phi_j(x) > 0$ for $x > \tilde{p}_j$. Thus, $A_{m,j}(x)$ is strictly decreasing with respect to x for $x > \tilde{p}_j$. Thus $A_{m,j}(\tilde{p}_m - \epsilon) > A_{m,j}(\tilde{p}_m)$. Using $u_j(\tilde{p}_m) = u_j(\tilde{p}_m - \epsilon)$, we can conclude that $u_m(\tilde{p}_m) = u_{m,max} < u_m(\tilde{p}_m - \epsilon)$. This contradicts with \tilde{p}_m belonging to the support set of $\Phi_m(\cdot)$. The result follows. ■

Lemma 6. $\tilde{p}_i \notin [\tilde{p}_m, \tilde{p}_{m-1})$ for $i \in \{1, \dots, m-2\}$.

To prove this, we use a contradiction argument. Suppose that there exists $\tilde{p}_j \in [\tilde{p}_m, \tilde{p}_{m-1})$ such that $j \in \{1, \dots, m-2\}$. We will prove that no $x \in (\tilde{p}_j, \tilde{p}_{m-1}]$ is in the support of $\Phi_m(\cdot)$. Thus there exists $u \in \{1, \dots, m-2\}$ such that \tilde{p}_{m-1} is in the support set of $\Phi_u(\cdot)$. We prove that the payoff of the seller when she offers u units with price $\tilde{p}_{m-1} + \epsilon$ is strictly greater than the payoff when offering with price \tilde{p}_{m-1} . This is in contradiction with \tilde{p}_{m-1} being the best response for player with availability u . The lemma follows.

Proof: The lemma follows by vacuity if $m \leq 2$. Take $m > 2$. Note that $\tilde{p}_{m-1} < v$. If not there is a jump of size 1 at price v when the seller offers $m-1$ units. Since $2m-2 > d = m$ for $m > 2$, using Lemma 1, $u_{m-1}(v - \epsilon) > u_{m-1}(v)$ for ϵ small enough. This is in contradiction with assigning a positive probability to price v in the equilibrium when seller offers $m-1$ units. Thus $\tilde{p}_{m-1} < v$.

Suppose there exists $\tilde{p}_j \in [\tilde{p}_m, \tilde{p}_{m-1})$ such that $j \in \{1, \dots, m-2\}$. We will prove that no $x \in (\tilde{p}_j, \tilde{p}_{m-1}]$ is in the support of $\Phi_m(\cdot)$. Thus (using this and Theorem 6), there exists $u \in \{1, \dots, m-2\}$ such that \tilde{p}_{m-1} is in the support set of $\Phi_u(\cdot)$. Consider $B_{m-1,u}(x)$ which is the summation of weighted distributions $\Phi_i(x)$ when $i \in \{2, \dots, m-1\}$. Thus, the distribution $\Phi_{m-1}(\cdot)$ is included in the summation of $B_{m-1,u}(x)$ (note that $m > 2$), and its coefficient is negative (Note that $d - (m-1) = 1 > 0$). Thus, $A_{m-1,u}(x)$ is strictly decreasing with respect to x for $x > \tilde{p}_{m-1}$. Thus $A_{m-1,u}(\tilde{p}_{m-1} + \epsilon) < A_{m-1,u}(\tilde{p}_{m-1})$. Using $u_{m-1}(\tilde{p}_{m-1}) = u_{m-1}(\tilde{p}_{m-1} + \epsilon)$, we can conclude that $u_u(\tilde{p}_{m-1}) = u_{u,max} < u_u(\tilde{p}_{m-1} + \epsilon)$. This is in contradiction with \tilde{p}_{m-1} being the best response for player with availability u . Note that $\tilde{p}_{m-1} < v$, and every price less than v which belongs to the support set of a distribution $\Phi_i(\cdot)$ should be a best response for players when offering i units. The lemma follows.

Now we complete the proof by proving that no $x \in (\tilde{p}_j, \tilde{p}_{m-1}]$ is in the support of $\Phi_m(\cdot)$. Suppose not. We will show that there exist an availability level f and two prices y_1 and y_2 , such that $\tilde{p}_j < y_1 < \tilde{p}_{m-1}$, belongs to the support set of $\Phi_m(\cdot)$, and both y_1 and y_2 belong to the support set of $\Phi_f(\cdot)$. Then we will show that $u_m(y_1) < u_m(y_2)$, which contradicts with y_1 being in the support set of $\Phi_m(\cdot)$.

Using the contradiction assumption, w is defined as,

$$w = \inf_{x \in (\tilde{p}_j, \tilde{p}_{m-1}] \text{ \& } x \in \text{Supp}(\Phi_m(\cdot))} x$$

Note that w is in the support set of $\Phi_m(\cdot)$. Now consider two cases:

- 1) $w > \tilde{p}_j$: In this case, using the definition of support sets, Theorem 6 (contiguity) and continuity, there exist ϵ and $f \in \{1, \dots, m-2\}$ such that w and $w - \epsilon$ belong to the support set of $\Phi_f(\cdot)$.

Take $y_1 = w$ and $y_2 = w - \epsilon$.

- 2) $w = \tilde{p}_j$: In this case, using continuity and the definition of infimum, there exists ϵ such that every $w + \epsilon$ belong to the support set of $\Phi_m(\cdot)$ and $\Phi_j(\cdot)$. Take $f = j$, $y_1 = w + \epsilon$, and $y_2 = w$.

Next, we will prove that $u_m(y_1) < u_m(y_2)$, which contradicts with y_1 being in the support set of $\Phi_m(\cdot)$. Note that $y_1 < v$, and every price less than v which belongs to the support set of a distribution $\Phi_i(\cdot)$ should be a best response for players when offering i units. This completes the proof.

Consider $B_{m,f}(x)$ which is the summation of weighted distributions $\Phi_i(x)$ when $i \in \{1, \dots, m-1\}$. Thus, the distribution $\Phi_f(\cdot)$ is included in the summation of $B_{m,f}(x)$, and its coefficient is negative (Note that $d - f > 0$). Thus, $A_{m,f}(x)$ is strictly decreasing with respect to x for $x \geq \tilde{p}_f$. Thus $A_{m,f}(y_2) > A_{m,f}(y_1)$. Using $u_f(y_1) = u_f(y_2)$, we can conclude that $u_m(y_1) < u_m(y_2)$. The contradiction argument is complete. ■

Therefore we established the ordering for $\Phi_m(\cdot)$ and $\Phi_{m-1}(\cdot)$. Now we are set to establish the ordering for the remaining support sets $\Phi_j(\cdot)$ for $j \in \{1, \dots, m-2\}$, knowing that for them $\tilde{p}_j \geq \tilde{p}_{m-1}$. The next is the counterpart of the Theorem 11 in symmetric setting. Using this proof, we can establish the ordering and disjoint property for the support sets.

Theorem 11. *The support set of $\Phi_l(\cdot)$ is a subset of $[\tilde{p}, \tilde{p}_j] \cup [v]$ for all integers $j \in [1, l]$.*

Proof: Consider support sets of $\Phi_j(\cdot)$, $\Phi_l(\cdot)$, and $j < l$. We will show that $u_l(a) < u_l(\tilde{p}_j)$ for all $a \in (\tilde{p}_j, v)$. Thus, no $a \in (\tilde{p}_j, v)$ is a best response for the seller with availability of l units. Therefore, the support set of $\Phi_l(\cdot)$ is a subset of $[\tilde{p}, \tilde{p}_j] \cup [v]$.

We now complete the proof, by showing that $u_l(a) < u_l(\tilde{p}_j)$ for all $a \in (\tilde{p}_j, v)$:

$$u_l(a) - \frac{l}{j}u_j(a) = A_{l,j}(a)$$

Note that if $\tilde{p}_j \geq v$, theorem follows by vacuity. Now we consider $\tilde{p}_j < v$. Since $j < l \leq m$, $j \leq m-1$. By Lemma 6, $\tilde{p}_{m-1} \leq \tilde{p}_j < a < v$, by Lemma 4, $A_{l,j}(a)$ is decreasing function of a for $a \in [\tilde{p}_{m-1}, v]$. Thus, $A_{l,j}(a) < A_{l,j}(\tilde{p}_j)$ for $a \in (\tilde{p}_j, v)$. On the other hand $u_j(a) \leq u_j(\tilde{p}_j)$ for all $a > \tilde{p}_j$, since \tilde{p}_j is a best response of a seller with availability j , therefore $u_l(\tilde{p}_j) > u_l(a)$. ■

Now we will prove the Theorem 10:

Proof: Note that the first place that we used the condition $d > \max\{m_1, m_2\}$ (in symmetric setting $d > m$) instead of $d = \max\{m_1, m_2\}$ ($d = m$) was in Section III-E. Thus all of the results before that apply also to the case that $d = m$. Theorem 11 provides exactly the same property in the Theorem 7 for the symmetric scenario. Thus the corollaries after the theorem follows. In addition, results in the

Section III-F follows, since they are based on results before the Section III-E and Theorem 7 and its corollaries. Thus Theorem 1 goes through in the case of a symmetric NE and $d = m$. The result follows. ■

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